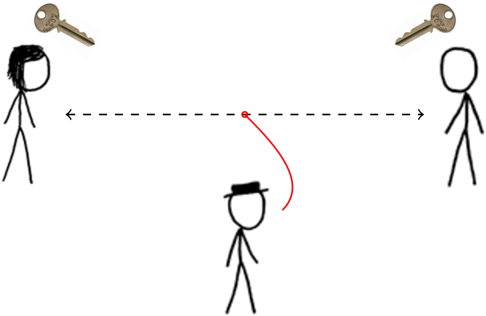
Post-Quantum Cryptography

Chris Peikert
University of Michigan

Tutorial, QIP 2022
6 March

Public-Key Cryptography

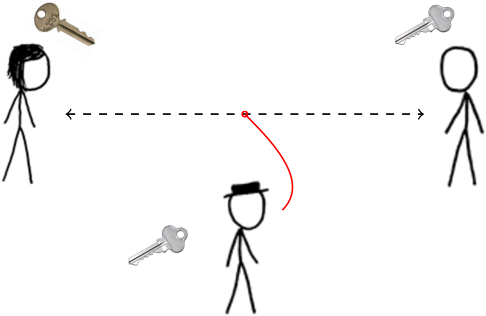
- Cryptography since the ancients: Alice, Bob need the same secret key



(Images courtesy xkcd.org)

Public-Key Cryptography

- A paradigm shift [Merkle'74,DH'76,RSA'77]: 'public-key' cryptography



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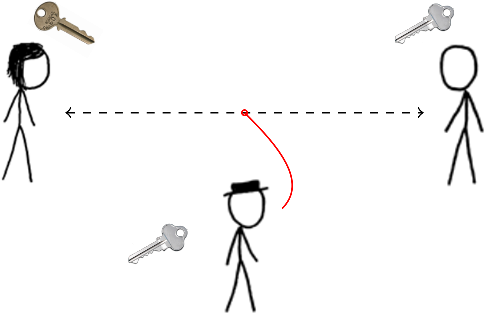

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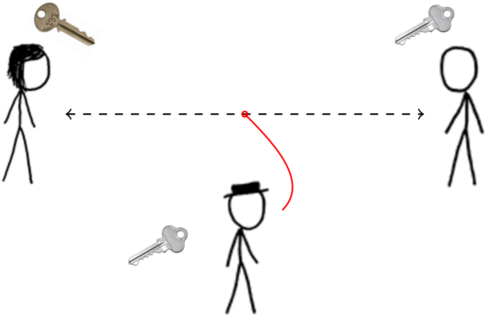
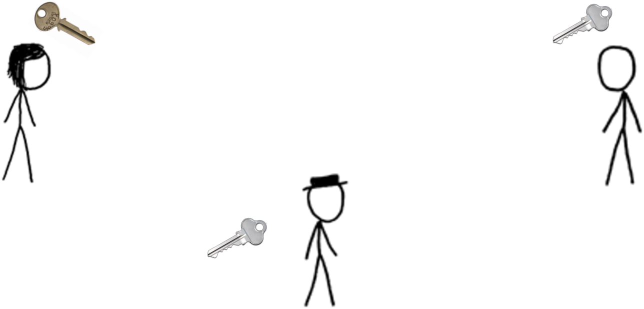


- Alice creates (related) public key 🔑 and secret key 🗝:
  - ★ Anyone can do 'public' ops using 🔑: encrypt, check authenticity
  - ★ Only Alice can do 'privileged' ops using 🗝: decrypt, attest

(Images courtesy xkcd.org)

Bread and Butter of PKC: Encryption



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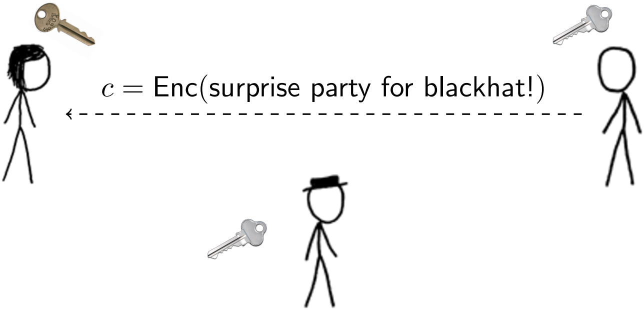

$c =$ Enc(surprise party for blackhat!)

- Alice can use the secret key to decrypt the message.

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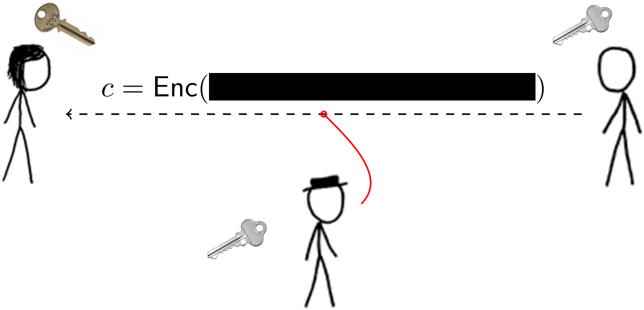
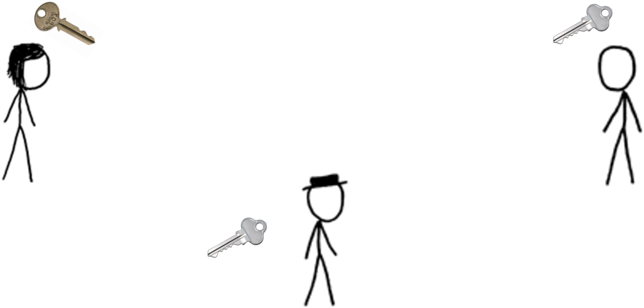


$c = \mathsf{Enc}($[redacted]$)$

- Alice can use the secret key to decrypt the message.
- Eavesdropper who gets the public key and ciphertext learns nothing about the message.

Bread and Butter of PKC: Digital Signatures



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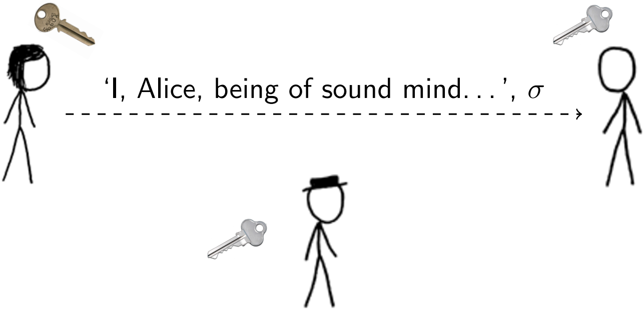
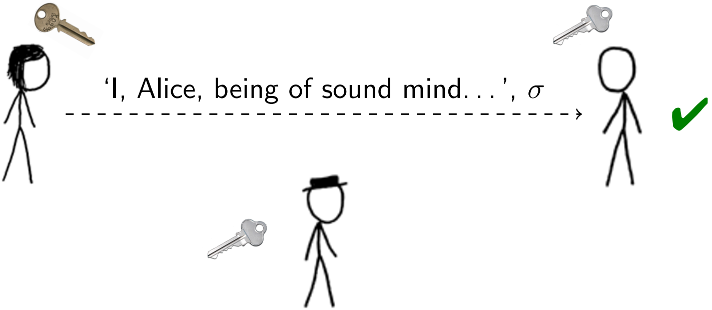


'I, Alice, being of sound mind. . . ', σ

- Alice uses her secret key to create a signature σ for a message.

Bread and Butter of PKC: Digital Signatures



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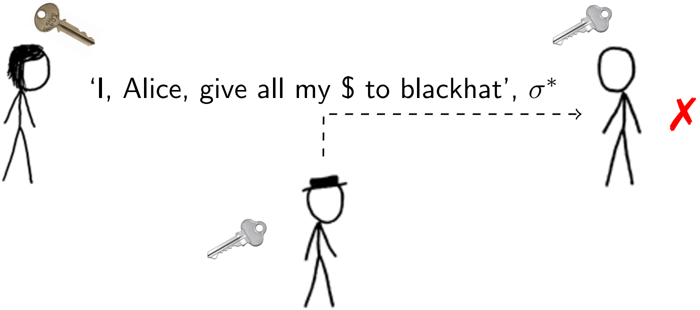

- Alice uses her secret key to create a signature σ for a message.
- Bob can use the public key to verify that the signature is authentic (for this specific message).
- Attacker can’t forge a valid signature σ^* for an unsigned message.

Hard Problems and PKC

- Public-key crypto inherently requires hard computational problems.

  For one: must be hard to compute the secret key from the public key.

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2. RSA/DH are 'based on' the hardness of factoring/dlog variants:
   Breaking RSA is not (much) easier than the 'RSA problem.' Trickier!

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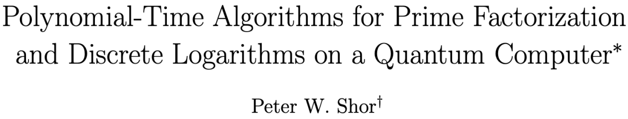

Polynomial-Time Algorithms for Prime Factorization
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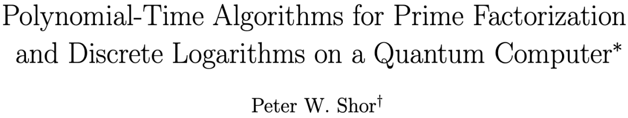

Polynomial-Time Algorithms for Prime Factorization
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- With a large-scale QC, [Shor’94] totally breaks DH, RSA, and all other widely used public-key crypto!

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Post-Quantum Cryptography (a.k.a. 'Quantum Resistant', 'Quantum Safe', . . .)

Design cryptosystems that can

run on (today's) classical computers,

while being

secure against quantum attacks.

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–BTTF (1985)

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- NIST PQC standardization process (2016–):
  3rd round, finalists and alternates chosen, selections imminent

Tutorial Agenda

1. A highly selective tour of the PQC landscape:
   concepts, key techniques, theory and practice
2. A lot/some/very little of what I know a lot/some/very little about:
   lattices / isogenies / MQ and codes
3. Important problems that need more scrutiny from quantum experts!

Lattices

Lattice-Based Cryptography

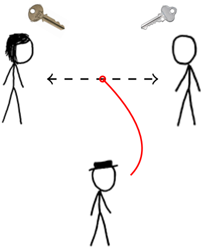
$y = g^x \bmod \mathrm{p}$

$N = p \cdot q$

$m^e \bmod N$

$e(g^a, g^b)$

$\Longrightarrow$



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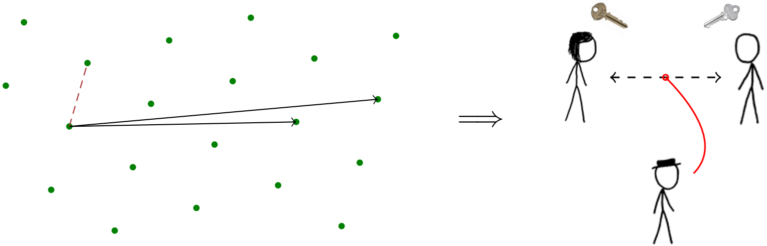

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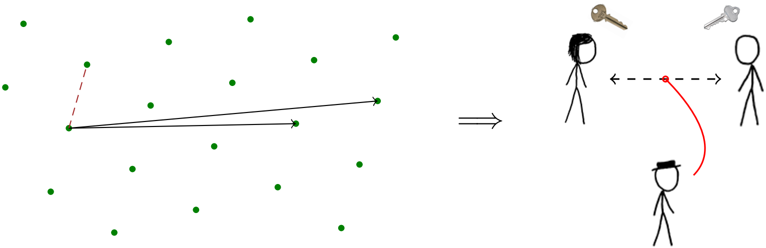

Why?

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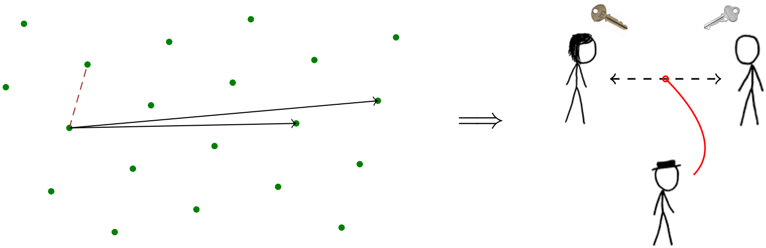

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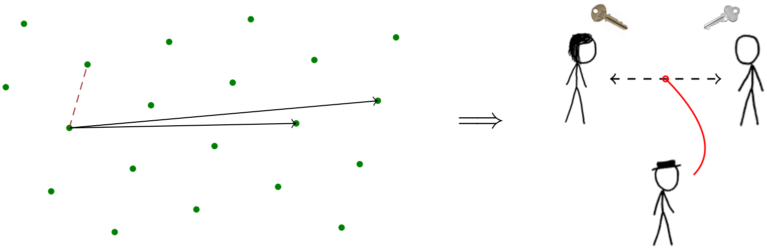

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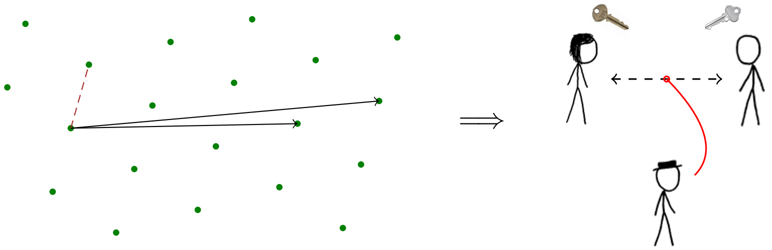
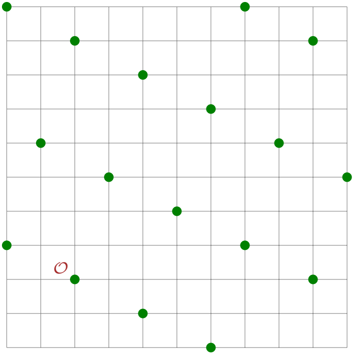


Why?

- Efficient: linear, embarrassingly parallel operations
- Resists quantum attacks (so far)
- Security from mild worst-case assumptions
- Solutions to ‘holy grail’ problems in crypto: FHE and related

What's a Lattice?

- A periodic 'grid' in $\mathbb{Z}^m$. (Formally: full-rank additive subgroup.)



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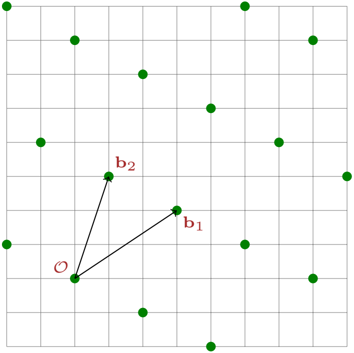

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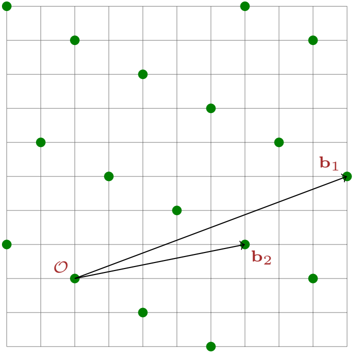

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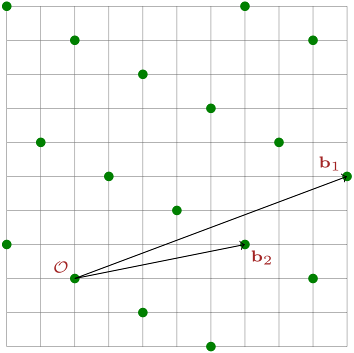

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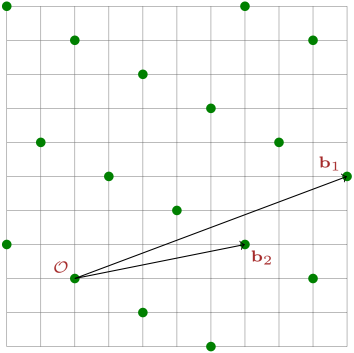

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Hard Lattice Problems

- Find/detect 'short' nonzero lattice vectors: (Gap)SVP$_\gamma$, SIVP$_\gamma$
- For $\gamma = \text{poly}(m)$, appears to require $2^{\Omega(m)}$ time and space, even quantumly. [LLL'82,Schnorr'87,...,AKS'01,...]

Lattices

Foundations, Digital Signatures

A Hard Problem: Short Integer Solution [Ajtai'96]

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A Hard Problem: Short Integer Solution [Ajtai'96]

- $\mathbb{Z}_q^n = n$-dimensional integer vectors modulo q
- Goal: find nontrivial $z_1, \ldots, z_m \in \{0, \pm 1\}$ such that:

$$z_1 \cdot \begin{pmatrix} | \\ \mathbf{a}_1 \\ | \end{pmatrix} + z_2 \cdot \begin{pmatrix} | \\ \mathbf{a}_2 \\ | \end{pmatrix} + \cdots + z_m \cdot \begin{pmatrix} | \\ \mathbf{a}_m \\ | \end{pmatrix} = \begin{pmatrix} | \\ 0 \\ | \end{pmatrix} \in \mathbb{Z}_q^n$$

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Collision-Resistant Hash Function

- Set $m > n \log_2 q$. Define 'compressing' $f_{\mathbf{A}} : \{0,1\}^m \to \mathbb{Z}_q^n$

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… yields short solution $\mathbf{z} = \mathbf{x} - \mathbf{x}' \in \{0, \pm 1\}^m$.

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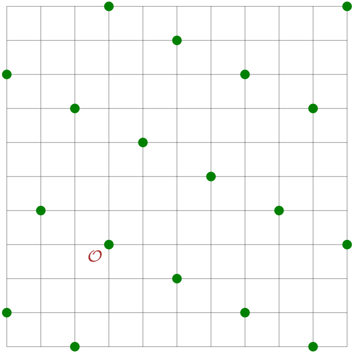

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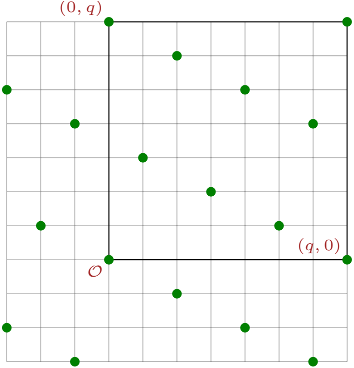

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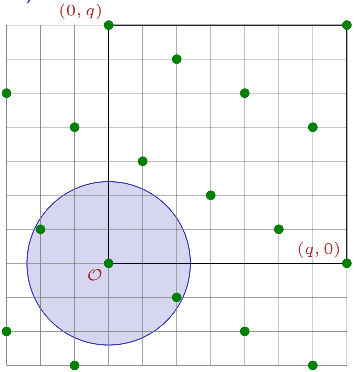
- 'Short' solutions $\mathbf{z}$ lie in ○



Cool! (But what does this have to do with lattices?)

- $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ defines a 'q-ary' lattice:

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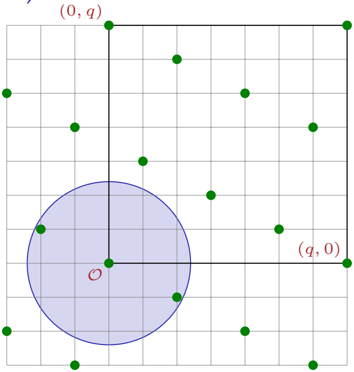

Worst-Case to Average-Case Reduction [Ajtai'96,...]

Finding 'short' ($\|\mathbf{z}\| \leq \beta \ll q$) nonzero $\mathbf{z} \in \mathcal{L}^{\perp}(\mathbf{A})$

(for uniformly random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$)

$\Downarrow$

solving GapSVP$_{\beta\sqrt{n}}$ and SIVP$_{\beta\sqrt{n}}$ on any n-dim lattice

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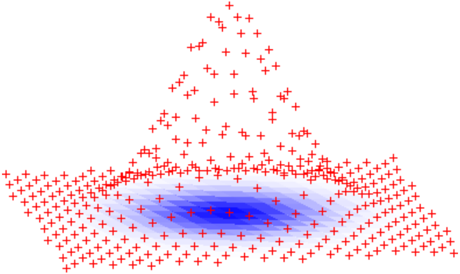

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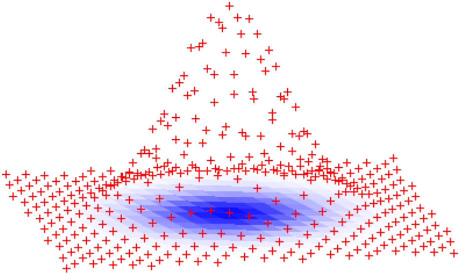

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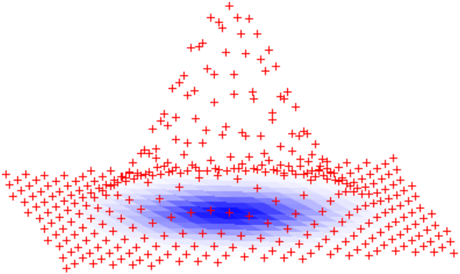

- Verify$(\mathbf{A}, \mu, \mathbf{z})$: check that $\mathbf{A}\mathbf{z} = H(\mu)$ and $\mathbf{z}$ is sufficiently short.
- Security: forging a signature for a new message μ^* requires finding short $\mathbf{z}^*$ s.t. $\mathbf{A}\mathbf{z}^* = H(\mu^*)$. This is SIS: hard!

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Questions

1. Is SIS (quantumly) hard for solution norm $\lesssim q$ in ℓ_∞ norm?
2. Tighter security reduction in QROM, or exploit looseness?

See [BDF+'12,KLS'18,DFMS'19,LZ'19].

Lattices

Public-Key Encryption

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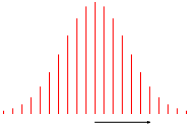

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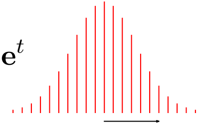

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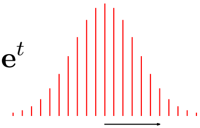

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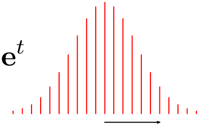

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(n/α)-approx *worst case* lattice problems $\underset{\uparrow}{\leq}$ search-LWE $\underset{\uparrow}{\leq}$ decision-LWE $\leq$ crypto

(quantum [R'05]) [BFKL'93,R'05,. . .]

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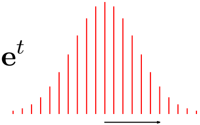

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- Also fully *classical* reductions, for worse params [Peikert'09,BLPRS'13]

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- LWE is ‘dual’ to SIS. Let

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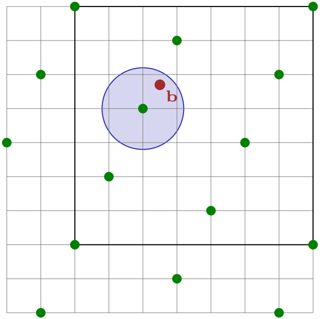

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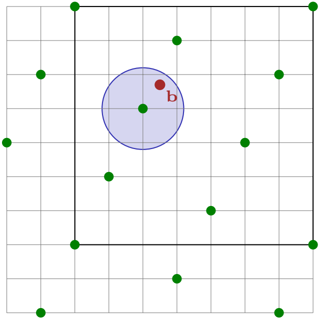

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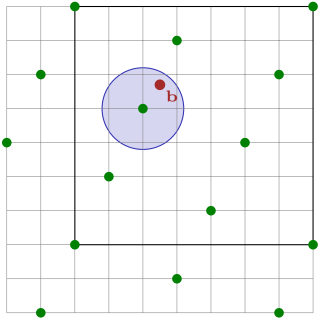

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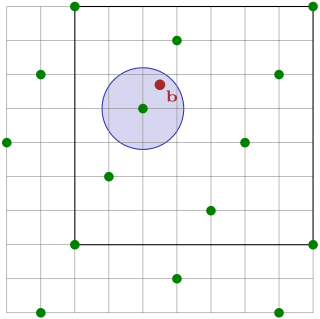

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Theorem [Regev’05]

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- Key Open Problem: ‘dequantize’ this theorem!

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- !!! Fully Homomorphic Encryption
- !!! Attribute-Based Encryption for arbitrary access policies

and much, much more. . .

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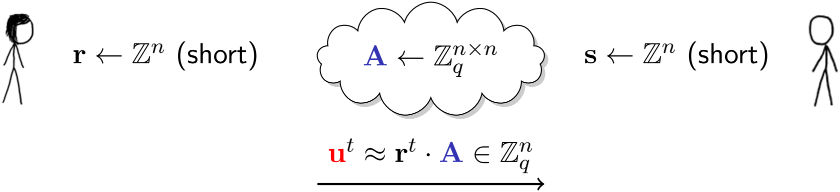

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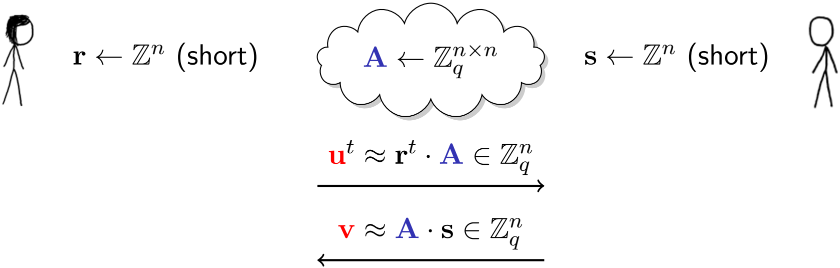

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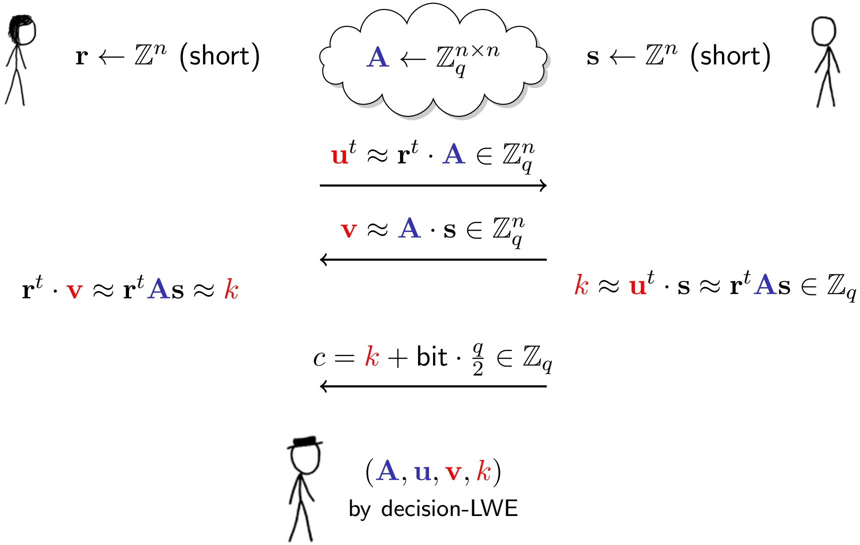

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$k \approx \mathbf{u}^t \cdot \mathbf{s} \approx \mathbf{r}^t \mathbf{A} \mathbf{s} \in \mathbb{Z}_q$

$$\xleftarrow{c = k + \text{bit} \cdot \frac{q}{2} \in \mathbb{Z}_q}$$

$(\mathbf{A}, \mathbf{u}, \mathbf{v}, k)$

by decision-LWE

LWE In Practice: Frodo(KEM) [BCD+'16,ABD+'17]

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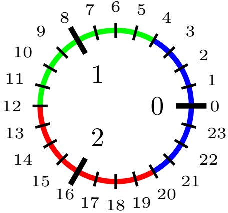
Learning With **Rounding** (LWR) [BanerjeePeikertRosen'12]



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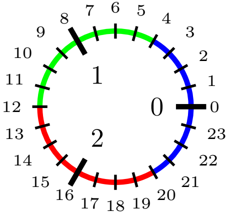

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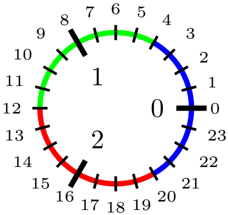

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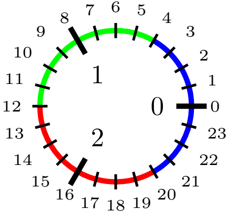

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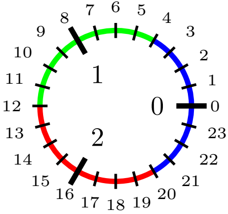

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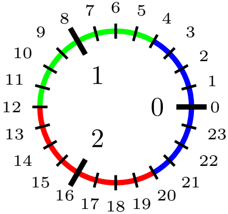

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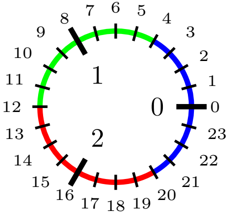

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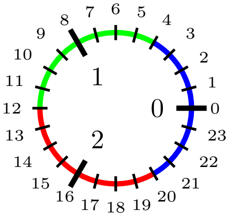

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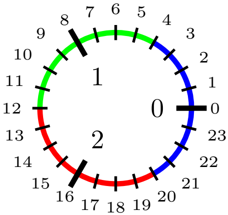

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Proof idea: w.h.p., $(\, \mathbf{a}_i \, , \, \lfloor \langle \mathbf{s}, \mathbf{a}_i \rangle + e \rceil_p \,) = (\, \mathbf{a}_i \, , \, \lfloor \langle \mathbf{s}, \mathbf{a}_i \rangle \rceil_p \,)$.

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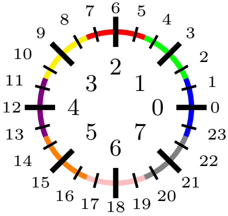

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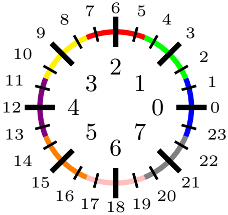

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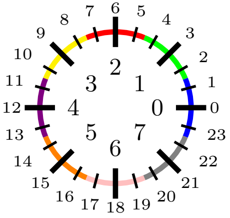

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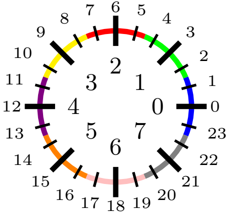

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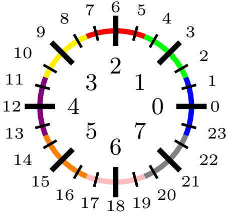

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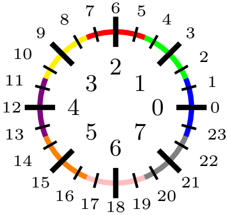

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Regev'02 uses rounding to quantumly reduce BDD to a 'noisy' cyclic hidden-shift problem, which has a $\exp(\sqrt{\log |G|})$ quantum algorithm.

Could those techniques be useful here?

Lattices

Efficiency from Algebraic Structure

SIS/LWE/LWR are Efficient(-ish)

$$(\cdots\ \mathbf{a}_i\ \cdots)\begin{pmatrix}\vdots\\ \mathbf{s}\\ \vdots\end{pmatrix}+e_i\ =\ b_i\in\mathbb{Z}_q$$

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- Inherently $\geq n^2$ time to encrypt & decrypt an n-bit message.

Wishful Thinking…

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- ⋆ Each eq. is d related eq.'s on a secret of dim $n = kd$ over $\mathbb{Z}_q$.

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⋆ Each eq. is d related eq.'s on a secret of dim $n = kd$ over $\mathbb{Z}_q$.

⋆ LWE: $d = 1, k = n$.
Ring-LWE: $d = n, k = 1$.
Module-LWE: interpolate.

LWE Over Rings/Modules, Over Simplified [LPR'10,BGV'11,LS'12]

- Let $\boxed{R = \mathbb{Z}[X]/(X^d + 1)}$ for d a power of two, and $\boxed{R_q = R/qR}$
  - ⋆ Elements of R_q are degree $< d$ polynomials with mod-q coefficients
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- **Decision**: distinguish $(\mathbf{a}_i \ , \ b_i)$ from uniform $(\mathbf{a}_i \ , \ b_i) \in R_q^k \times R_q$

Hardness of Ring/Module-LWE

Theorems [. . . ,SSTX'09,LPR'10,LS'12,PRS'17,RSW'18,. . .]

worst-case approx-SVP on rank-k *module* lattices over R $\leq$ search R^k-LWE $\leq$ decision R^k-LWE

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3. Are there reverse reductions? (Seems not, without increasing k. . .)

Ring/Module-LWE In Practice: Kyber, SABER, NTRU(')

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- Theorems require moderate error sizes $\gg \sqrt{n}$ in each coefficient.

  Systems use small error sizes $\in [1, 7]$.

  Seems hard according to cryptanalysis. Theory? (Quantum) attacks?

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3. Many important questions need attention from quantum experts. The future of our digital security may depend on it!

Bonus: Isogenies

Elliptic Curves and Isogenies

- An elliptic curve E over a field $\mathbb{F}$ is the set of solutions $(x, y) \in \mathbb{F}^2$ to

  $$y^2 = x^3 + ax + b$$

  for suitable fixed $a, b \in \mathbb{F}$, plus a 'point at infinity' $\mathcal{O}$.

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- There are proposals to use conjectured-hard problems related to finding isogenies between isogenous curves.

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   No (quantum) cryptanalytic improvements since original proposal. Opportunity?

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- Signatures [Stolbunov'12,DeFeoGalbraith'19,BeullensKleinjungVercauteren'19]: pk + sig = 1468 bytes at same claimed security level

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E.g., $\log(\text{queries}) \cdot \log(\text{QRACM}) \gtrsim n$.

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*Independently, [BonnetainSchrottenloher'20] gave a complementary, theoretical c-sieve analysis, arriving at similar conclusions.

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Thanks!