The Geometry of Rings

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$$\|e+f\| \leq \|e\| + \|f\| \qquad \|e \cdot f\| \leq \sqrt{n} \cdot \|e\| \cdot \|f\|.$$

"Expansion factor" $\sqrt{n}$ is worst-case. ("On average," $\approx \sqrt{\log n}$.)

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$$c_1 \leftarrow R_q \quad \text{and} \quad c_0 = -c_1 \cdot s + e \in R_q$$

and output $c(S) = c_0 + c_1 S \in R_q[S]$. (Notice: $c(s) = e \bmod q$.)

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  Many mults $\Rightarrow$ large power of expansion factor $\Rightarrow$ tiny error rate $\alpha \Rightarrow$ big parameters!

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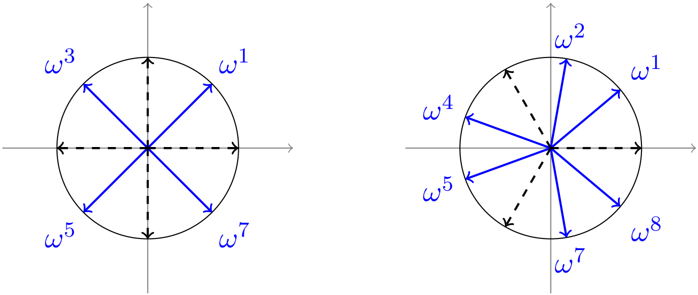

$\Phi_8(X) = 1 + X^4$ $\qquad$ $\Phi_9(X) = 1 + X^3 + X^6$

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✗ Provable hardness also degrades with expansion factor: pay twice!

Talk Agenda

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   - ✔ No expansion factor anywhere
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Based on:

LPR'10 V. Lyubashevsky, C. Peikert, O. Regev.
"On Ideal Lattices and Learning with Errors Over Rings."

LPR'12 V. Lyubashevsky, C. Peikert, O. Regev.
"A Toolkit for Ring-LWE Cryptography."

Cyclotomic Rings

Key Facts

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  Notice!: tensor basis $\neq$ power basis $\{X^j\}$ for $0 \leq j < \varphi(m)$.

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- Bottom line: can reduce operations in R to independent operations in prime-power cyclotomic rings $\mathbb{Z}[X_i]/\Phi_{m_i}(X_i)$.

Canonical Geometry of R

- $R = \mathbb{Z}[X]/\Phi_m(X)$ has $n = \varphi(m)$ ring embeddings (homomorphisms) into $\mathbb{C}$, each given by evaluation at a root of Φ_m:

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- The *canonical embedding* σ of R into $\mathbb{C}^n$ is $\sigma(a) = \big(a(\omega^i) \big)_{i \in \mathbb{Z}_m^*}$.

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✔ Expansion is element-specific. No more ring "expansion factor."

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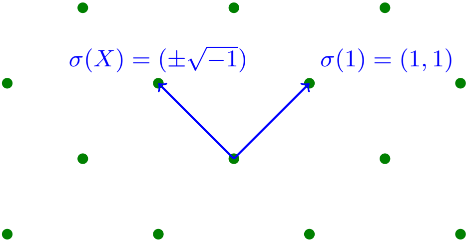

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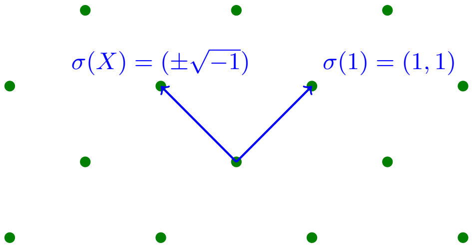

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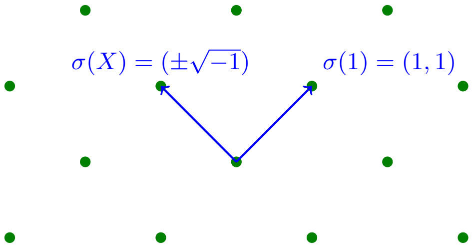

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✔ For any j, $\|X^j\|_2 = \sqrt{n}$ and $\|X^j\|_\infty = 1$.

✔ Power basis $\{1, X, \ldots, X^{n-1}\}$ is orthogonal under embedding σ.
So coefficient/canonical embeddings equivalent (up to $\sqrt{n}$ scaling).

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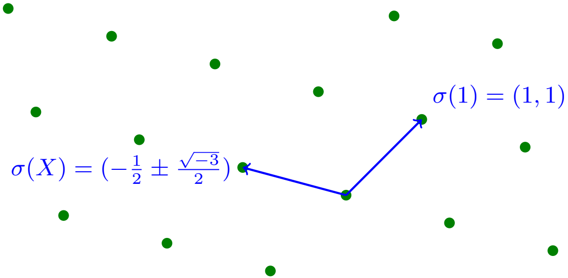

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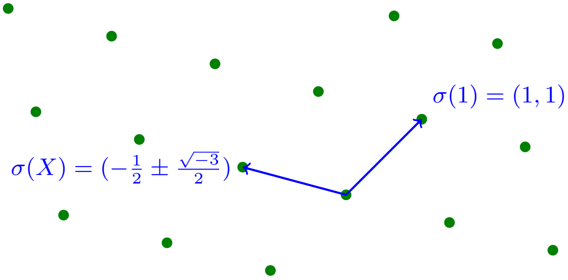

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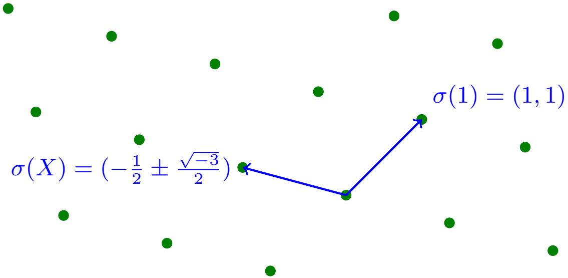

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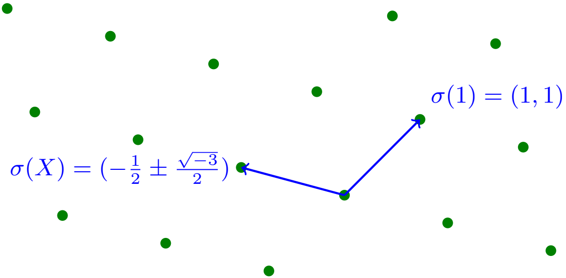

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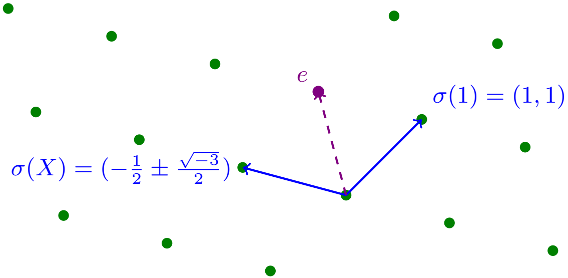

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  E.g., $\|e\| = \|1\| = \|X\| = \sqrt{n}$ but $e = 1 + X$.

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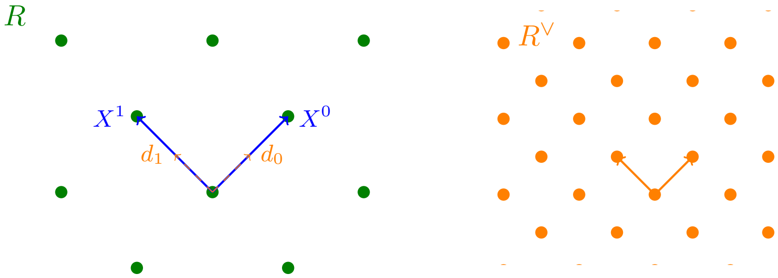

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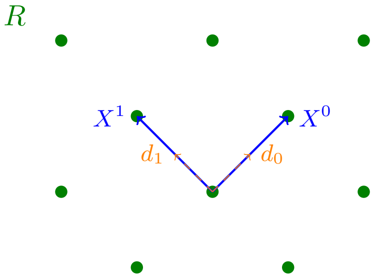

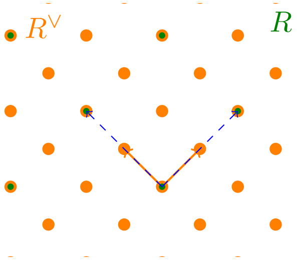

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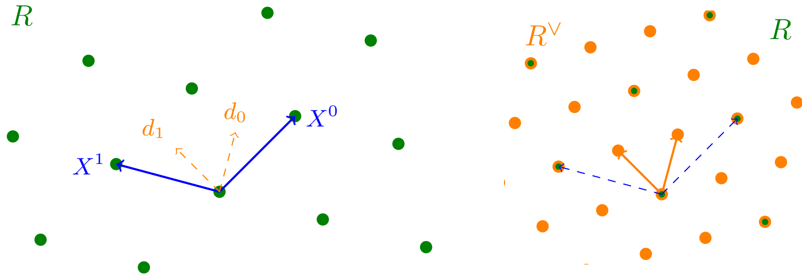

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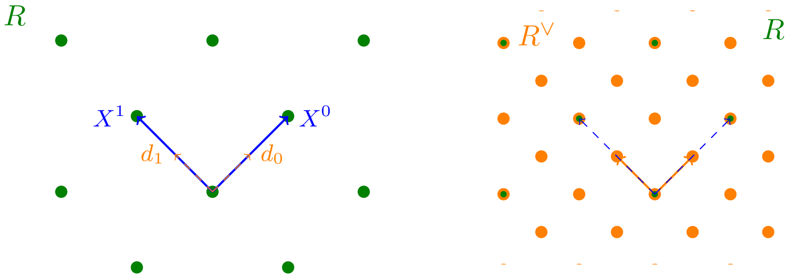

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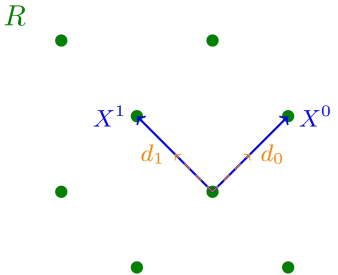

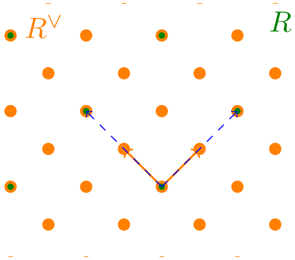

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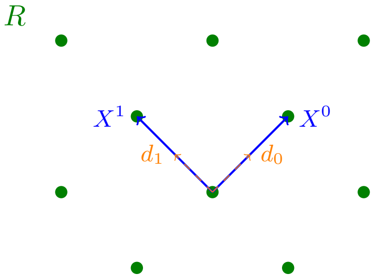

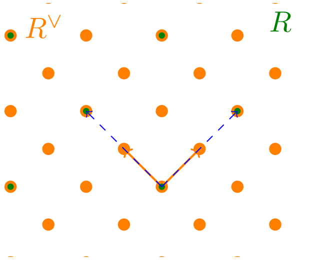

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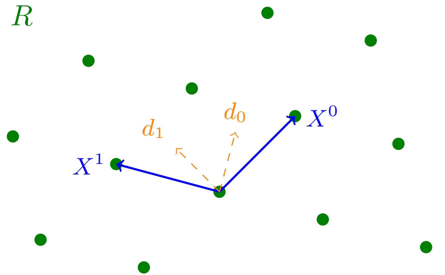

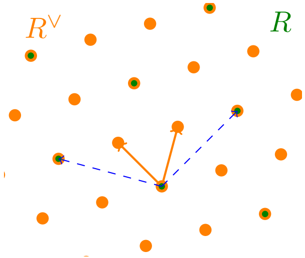

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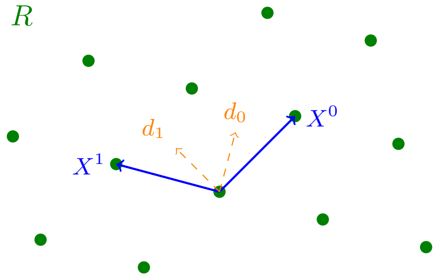

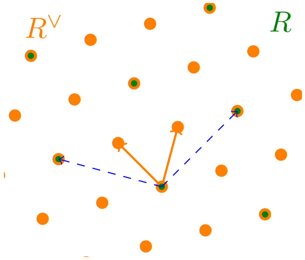

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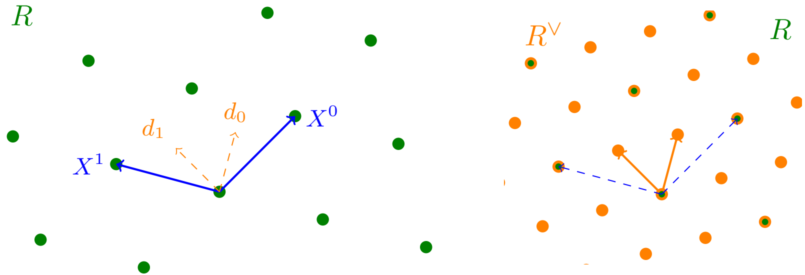

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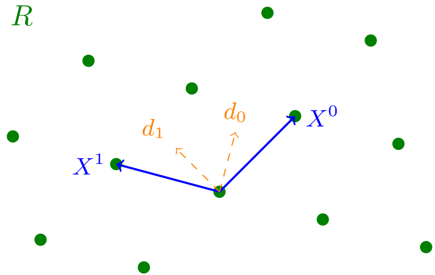

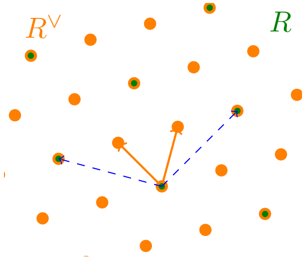

Ring-LWE: The Complete Definition [LPR'10]

Ring $\boxed{R := \mathbb{Z}[X]/\Phi_m(X)}$ for any m, $\quad R_q = R/qR,\ R_q^\vee = R^\vee/qR^\vee.$

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Theorem

For any m, ring-LWE with error std. dev. $\alpha q \geq 6^*$
is (quantumly) as hard as
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- EvalMul$(c, c') = (c \cdot c')(S) \in (R^\vee)_q^k[S]$ where $k = \deg(c) + \deg(c')$.

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  and output $c(S) = c_0 + c_1 S \in R_q^\vee[S]$. (Note: $c(s) = e \bmod qR^\vee$.)
- $\mathsf{Dec}_s(c(S))$: get short $d \in R^\vee$ s.t. $d = c(s) \bmod qR^\vee$.

  Correctness: $d = e$, if e's decoding basis $\mathbb{Z}$-coeffs $\in (-q/2, q/2)$.
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  - So need $q \approx 6^k m^{k-1} \sqrt{n} < (6m)^k$ to decrypt deg-k ciphertexts.
    Versus $q \approx \gamma^{k-1} n^k$ via expansion factor $\gamma \gg \sqrt{n}$.
    $\Rightarrow \approx \gamma^{k-1}$ factor improvement in error rate.

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