Faster Bootstrapping with Polynomial Error

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Fully Homomorphic Encryption [RAD'78,Gentry'09]

- FHE lets you do this:

$$\boxed{\mu} \longrightarrow \boxed{\mathsf{Eval}(f)} \longrightarrow \boxed{f(\mu)}$$

A cryptographic "holy grail" with countless applications.

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- Thus far, "bootstrapping" is required to achieve unbounded FHE.

Bootstrapping: SHE $\rightarrow$ FHE [Gentry'09]

- Homomorphically evaluates the SHE decryption function to "refresh" a ciphertext $\boxed{\mu}$, allowing further homomorphic operations.

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- Can we do better?

Bootstrapping with Polynomial Error [BrakerskiVaikuntanathan'14]

- Error growth for multiplication in [GSW'13] is asymmetric:

Error in $\mathbf{C} := \mathbf{C}_1 \boxdot \mathbf{C}_2$ is $\mathbf{e} := \mathbf{e}_1 \cdot \mathrm{poly}(\lambda) + \mu_1 \cdot \mathbf{e}_2$.

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- Make multiplication right-associative:

$$\mathbf{C}_1 \boxdot (\cdots (\mathbf{C}_{t-2} \boxdot (\mathbf{C}_{t-1} \boxdot \mathbf{C}_t)) \cdots) \text{ has error } \sum_i \mathbf{e}_i \cdot \mathsf{poly}(\lambda)$$

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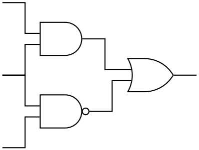
- Barrington's Theorem



$$\Rightarrow \quad \begin{array}{ccccc} (P_{0,1}) & (P_{1,1}) & \cdots & (P_{14,1}) & (P_{15,1}) \\ \\ (P_{0,0}) & (P_{1,0}) & \cdots & (P_{14,0}) & (P_{15,0}) \end{array}$$

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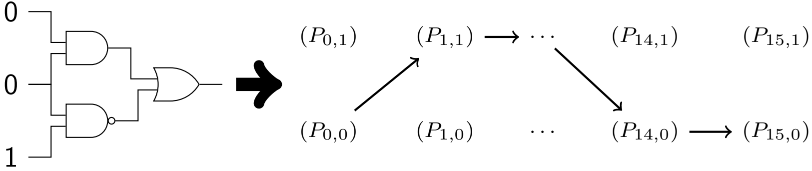

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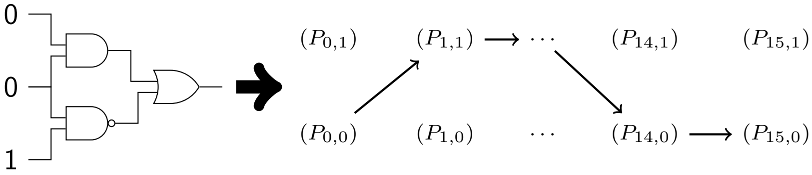

depth $d \approx 3 \log \lambda$

length $4^d \approx \lambda^6$

✗ Problem: Barrington's transformation is very inefficient.

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2. Variant of [GSW'13] encryption scheme
   - ★ Very simple description and error analysis
   - ★ Enjoys full re-randomization of error as a natural side effect

     Cf. [BV'14]: partial re-randomization, using extra key material

Simpler GSW Variant

- "Gadget" $\mathbb{Z}_q$-matrix $\mathbf{G}$ [MP'12]: for any $\mathbb{Z}_q$-matrix $\mathbf{A}$,

  $\mathbf{G}^{-1}(\mathbf{A})$ is short and $\mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{A}) = \mathbf{A} \pmod{q}$.

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- Old method [GSW'13]: $\mathbf{G}^{-1}$ is deterministic bit decomposition.
- New: $\mathbf{G}^{-1}$ samples a (random) subgaussian preimage.
  - $\Rightarrow$ Tight $O(\sqrt{n})$ error growth, full rerandomization of error

Overview of Our Bootstrapping Algorithm

- Decryption in LWE-based schemes can be expressed as

$$\mathsf{Dec}_{\mathbf{s}}(\mathbf{c}) := \lfloor \langle \mathbf{s}, \mathbf{c} \rangle \rceil_2 \in \{0,1\} \text{ with } \mathbf{s} \in \mathbb{Z}_q^n,\ \mathbf{c} \in \{0,1\}^n$$

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- Remains to implement $\boxplus$ and Equals for plaintext space $\mathbb{Z}_q$.

Warmup: Embedding $(\mathbb{Z}_q, +)$ into $(S_q, \cdot)$

$$
\begin{array}{ccccc}
\mathbb{Z}_q & \mathsf{0} & \mathsf{1} & \dots & q-1 \\
S_q &
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} &
\begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} &
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\begin{array}{ccccc}
\mathbb{Z}_q & 0 & 1 & \cdots & q-1 \\
S_q &
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\cdots &
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Thanks!