Unexpected Applications of Fully Homomorphic Encryption

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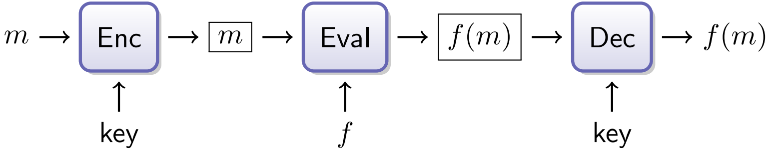
University of Michigan

Public Key Cryptography
8 May 2023

Fully Homomorphic Encryption [RAD'78,Gentry'09,...]

- FHE lets us do this:



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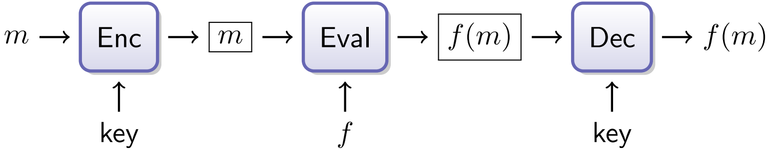

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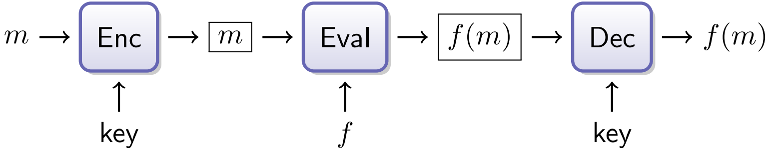

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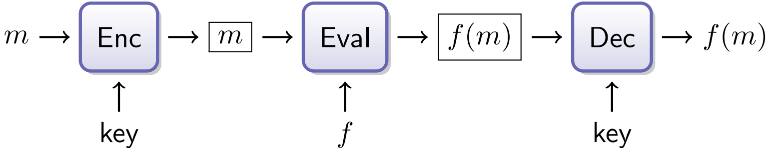

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Instead, *compactness* and *special structure* of FHE scheme are essential!

Background and the Central Equation

Homomorphic Computation [GentrySahaiWaters'13,...,deCastroP'23]

Theorem

- For *any* matrix $\mathbf{A}$ and (Boolean) function f, can compute $\mathbf{A}_f$. Then for *any* input x, can compute "short" matrix $\mathbf{S}_{f,x}$ satisfying

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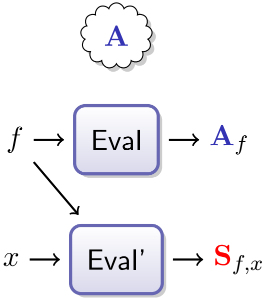

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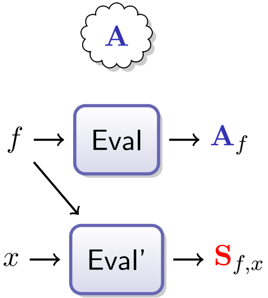

Implies LWE-Based FHE

- Ciphertext $\mathbf{A} = \mathbf{B} + \text{Encode}(x)$ where $\mathbf{sB} \approx \mathbf{0}$. Hides x by LWE.

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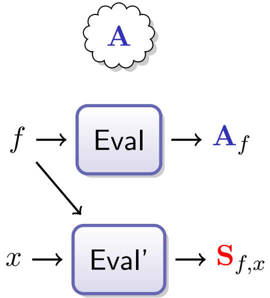

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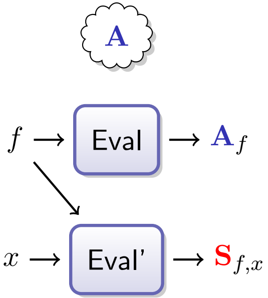

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- Decryption:

$$\mathbf{s}\mathbf{A}_f = \mathbf{s}\mathbf{B} \cdot \mathbf{S}_{f,x} + \mathbf{s} \cdot \mathsf{Encode}(f(x)) \approx \mathbf{s} \cdot \mathsf{Encode}(f(x)).$$

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- **Multiplication:** define $\mathbf{S}_{\times, x_1} = \left[\begin{smallmatrix} \mathbf{G}^{-1}(\mathbf{A}_2) \\ x_1 \mathbf{I} \end{smallmatrix}\right]$ and $\mathbf{A}_{\times} = \mathbf{A}_1 \cdot \mathbf{G}^{-1}(\mathbf{A}_2)$:

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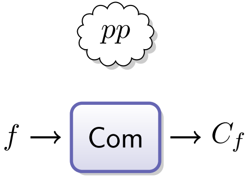
Functional Commitments

Functional Commitments [LibertRamannaYung'16]



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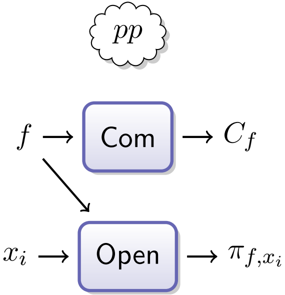

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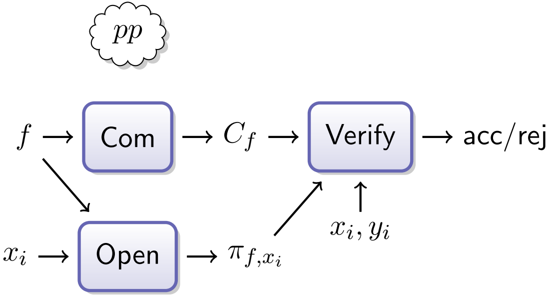

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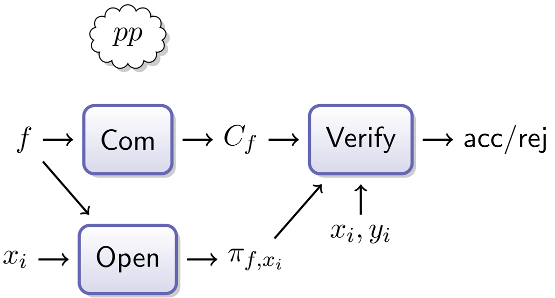

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Applications

- Specializations: vector/key-value/polynomial/linear commitments [LY'10,KZG'10,LRY'16,BBF'19]
- Verifiable outsourced storage/data structures [BGV'11,PSTY'13]
- Accumulators, updateable ZK sets/databases [BdM'93,MRK'03,Lis'05]
- Outsourced committed programs [GSW'23]
- And much more... [CPSZ'18,BFS'20,BDFG'21,...]

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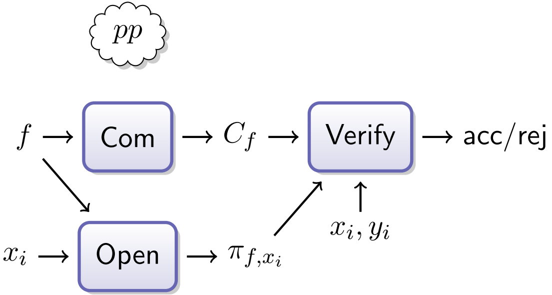

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Basic Security Properties

- **Evaluation binding:** infeasible to find $C^*, x^*, y_0^* \neq y_1^*, \pi_0^*, \pi_1^*$ s.t. Verify$(pp, C^*, x^*, y_b^*, \pi_b^*) =$ acc for $b \in \{0, 1\}$. (No hiding required!)

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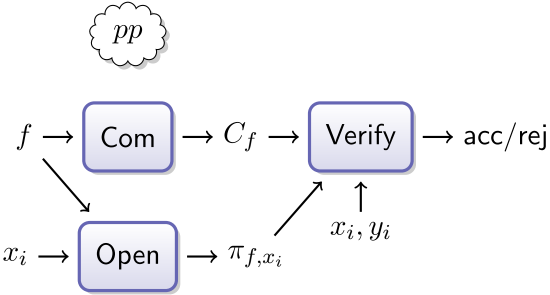

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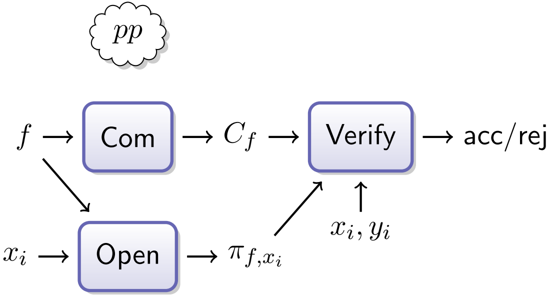

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- **Zero knowledge:** C_f and π_{f,x_i} reveal nothing except for $x_i, f(x_i)$.

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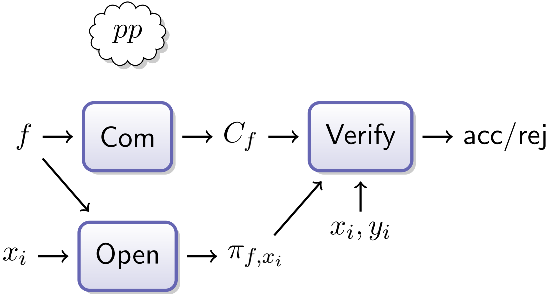

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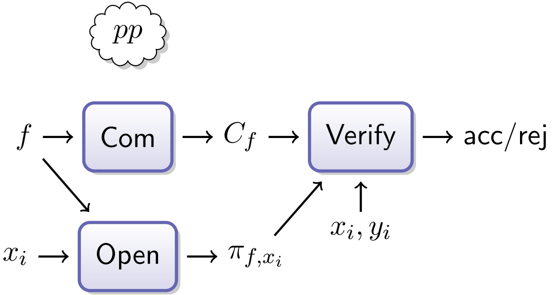

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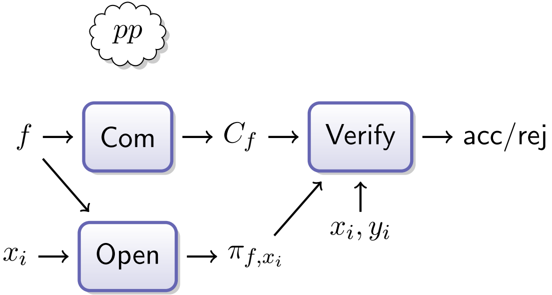

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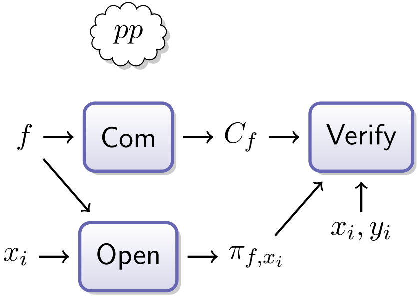

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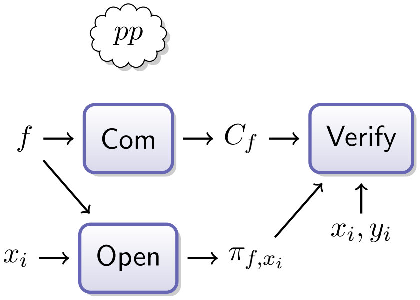

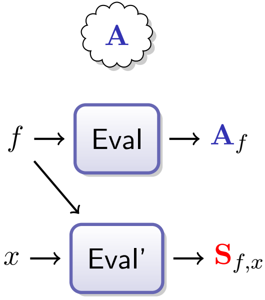

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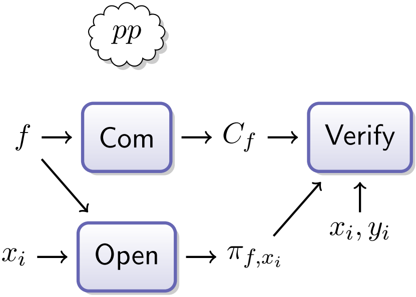

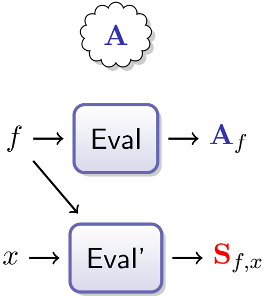

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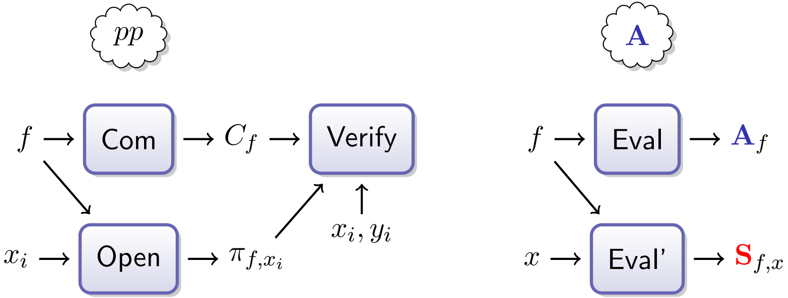

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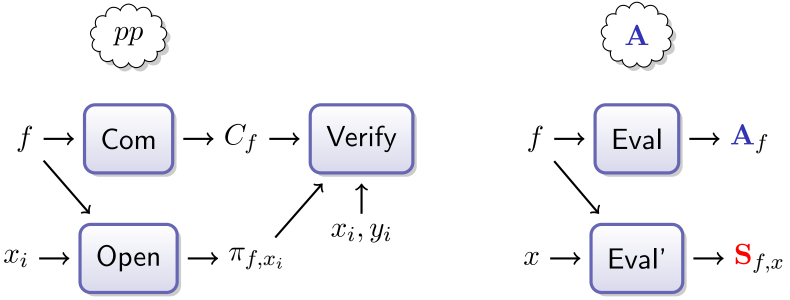

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- RHS has short nonzero column $\implies$ solves SIS for $\mathbf{A} - \mathrm{Encode}(x^*)$.

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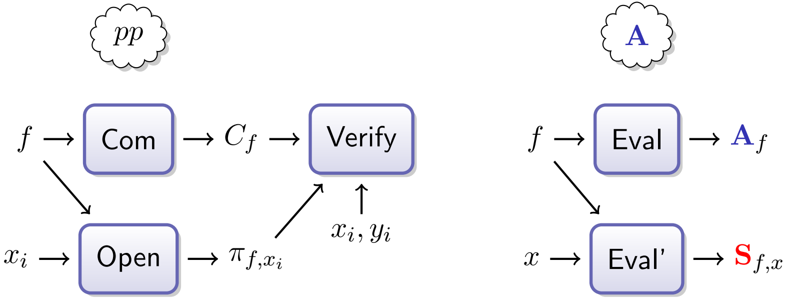

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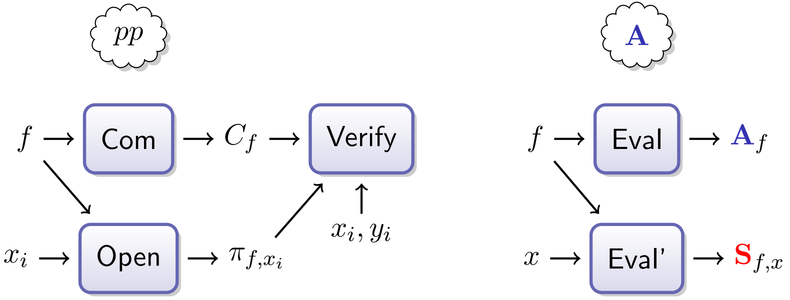

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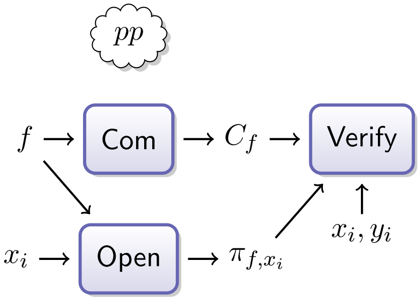

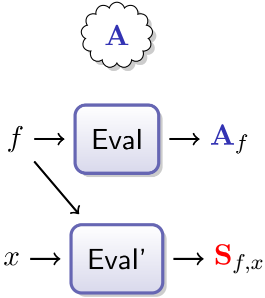

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- ZK (w/target binding) via Eval privacy and preimage sampling.

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- Unlike FHE, no hiding or 'structure' needed: public f and x, no sk, unstructured $pp = \mathbf{A}$.
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$$(\mathbf{A} - \text{Encode}(x)) \cdot \mathbf{S}_{f,x} = \mathbf{A}_f - \text{Encode}(f(x)).$$

- Similar ideas in [WeeWu'23] FCs, but:
  - ⋆ structured CRS (private-key setup);
  - ⋆ swapped Prove/Verify burden;
  - ⋆ smaller proofs;
  - ⋆ based on new, ad-hoc BASIS assumption.

Instantiating Fiat-Shamir and Noninteractive Zero Knowledge

(Noninteractive) Zero Knowledge [BlumDeSantisMicaliPersiano'88]

- Assuming OWFs, every NP language has a ZK proof/argument. [GoldreichMicaliWigderson'86,NguyenOngVadhan'06]

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- In 'plain' model, NIZK = BPP (trivial).

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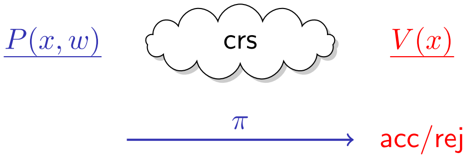

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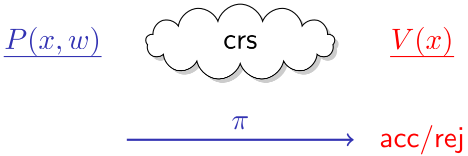

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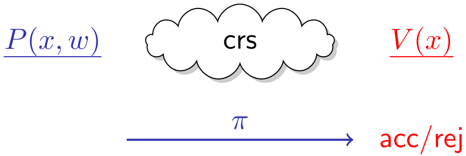

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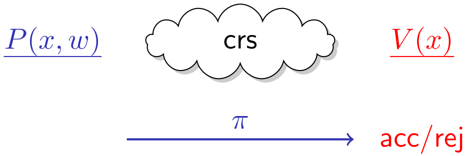

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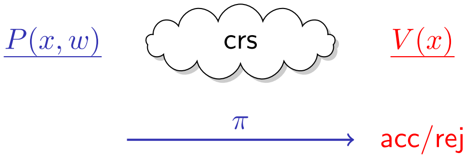

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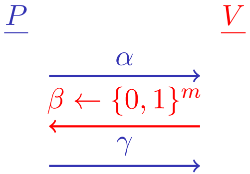

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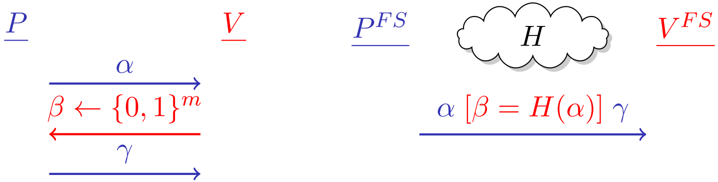

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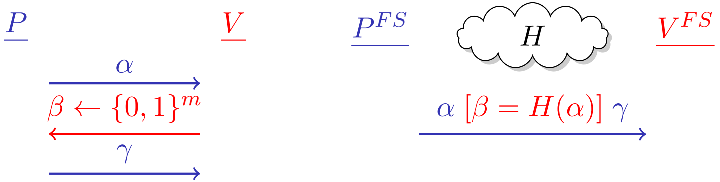

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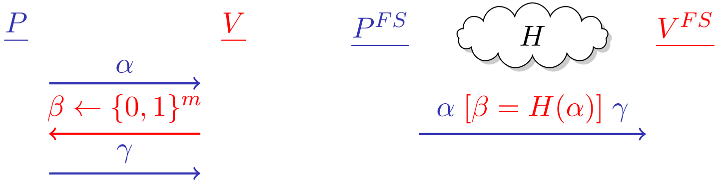

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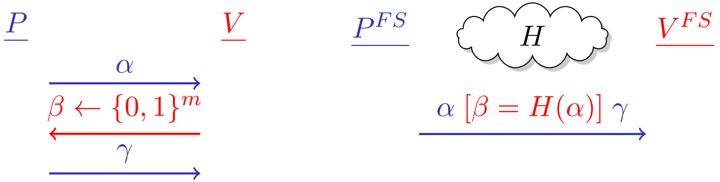

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2. Can a cheating P^* find such values, given H? (Proof vs. argument.)

Fiat-Shamir, Soundly [KRR'17,CCRR'18,HL'18,CCHLRRW'19]

$\underline{P^{FS}}$ H $\underline{V^{FS}}$

$\alpha\ [\beta = H(\alpha)]\ \gamma$ $\longrightarrow$

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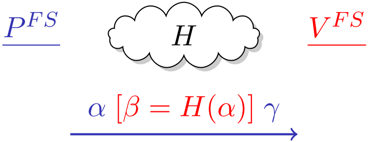

- A correlation-intractable [CGH'98] hash family $\mathcal{H}$ suffices:

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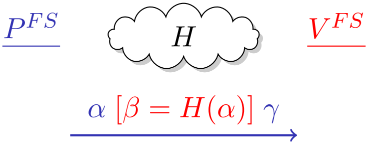

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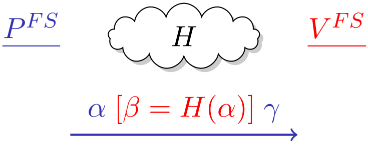

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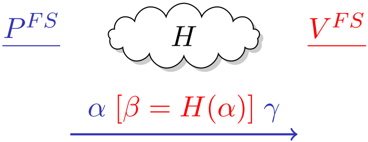

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  Such $\beta = C_{sk}(\alpha)$ using a trapdoor sk for decrypting α.

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  2. Statistical (LWE): over $H \leftarrow \mathcal{H}_C \overset{c}{\approx} \mathcal{H}$, such α do not exist w.h.p.
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Key Point: $\mathbf{a}_\alpha$ can 'hide' a circuit output $\mathbf{y}$ from the same domain, letting the two values 'mix'/cancel out.
Can reason about more than the hidden $\mathbf{y}$ alone.

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   (Tweak: can make $H(\alpha) = C(\alpha)$ impossible using LWE matrix $\mathbf{B}$.)

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  Breaking CI
  $\Rightarrow$ equating (public) hash value and (hidden) computed value
  $\Rightarrow$ cancellation solves SIS via Eval'.

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