Bootstrapping (with Small Error Growth)

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Fully Homomorphic Encryption [RAD'78,Gentry'09]

- FHE lets you do this:

$$\boxed{\mu} \longrightarrow \boxed{\mathsf{Eval}(f)} \longrightarrow \boxed{f(\mu)}$$

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- Thus far, "bootstrapping" is required to achieve unbounded FHE.

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  Error growth of Eval(Dec) determines strength of cryptographic assumption – e.g., initial LWE noise "rate" of $\boxed{sk}$.

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Can we do better??

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1. Branching program bootstrapping with (large) polynomial runtime and error growth [BrakerskiVaikuntanathan'14]
2. Arithmetic bootstrapping with small polynomial runtime and growth [Alperin-SheriffPeikert'14]
3. Fast (< 1s) ring-based implementation [DucasMicciancio'15]

Somewhat Homomorphic Encryption [GentrySahaiWaters'13]

- Recall "gadget" matrix $\mathbf{G}$ over $\mathbb{Z}_q$ [MP'12]: for any matrix $\mathbf{A}$ over $\mathbb{Z}_q$,

  $\mathbf{G}^{-1}(\mathbf{A})$ is short (over $\mathbb{Z}$) and $\mathbf{G} \cdot \mathbf{G}^{-1}(\mathbf{A}) = \mathbf{A} \pmod{q}$.

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- (Can randomize $\mathbf{G}^{-1}$ for tighter error growth, full rerandomization.)

Bootstrapping with Polynomial Error [BrakerskiVaikuntanathan'14]

- Error growth for multiplication is asymmetric and "quasi-additive:"

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  $\mathbf{C}_1 \boxdot (\cdots (\mathbf{C}_{t-2} \boxdot (\mathbf{C}_{t-1} \boxdot \mathbf{C}_t)) \cdots)$ has error $\sum_i \mathbf{e}_i \cdot \text{poly}(\lambda)$.

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- Generalizes to orthogonal matrices over $\mathbb{Z}$, e.g., permutation matrices. Encrypt bitwise:

$$\underbrace{\begin{pmatrix} \boxed{0} & \boxed{1} \\ \boxed{1} & \boxed{0} \end{pmatrix}}_{\mathbf{P}_1} \boxdot \underbrace{\begin{pmatrix} \boxed{0} & \boxed{1} \\ \boxed{1} & \boxed{0} \end{pmatrix}}_{\mathbf{P}_2} = \underbrace{\begin{pmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{pmatrix}}_{\mathbf{P}_1 \cdot \mathbf{P}_2}$$

$$\underbrace{\begin{pmatrix} \mathbf{e}_{1,1} & \mathbf{e}_{1,2} \\ \mathbf{e}_{2,1} & \mathbf{e}_{2,2} \end{pmatrix}}_{\mathbf{E}}, \underbrace{\begin{pmatrix} \mathbf{f}_{1,1} & \mathbf{f}_{1,2} \\ \mathbf{f}_{2,1} & \mathbf{f}_{2,2} \end{pmatrix}}_{\mathbf{F}} \to \mathbf{E} \cdot \mathrm{poly}(\lambda) + \underbrace{\begin{pmatrix} \mathbf{f}_{2,1} & \mathbf{f}_{2,2} \\ \mathbf{f}_{1,1} & \mathbf{f}_{1,2} \end{pmatrix}}_{\mathbf{P}_1 \cdot \mathbf{F}}$$

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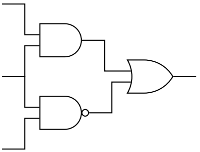

$$\begin{array}{ccccc} \mathbf{P}_{0,1} & \mathbf{P}_{1,1} & \cdots & \mathbf{P}_{14,1} & \mathbf{P}_{15,1} \\ \mathbf{P}_{0,0} & \mathbf{P}_{1,0} & \cdots & \mathbf{P}_{14,0} & \mathbf{P}_{15,0} \end{array}$$

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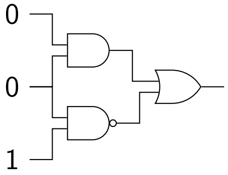

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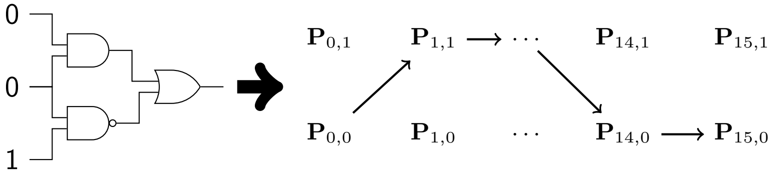

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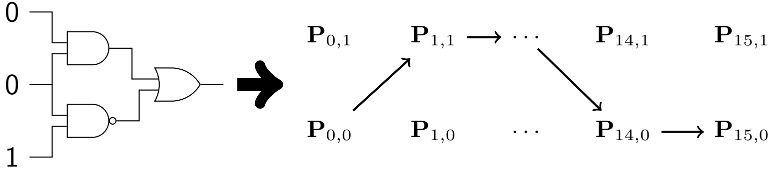

depth $d \approx 3 \log \lambda$ length $4^d \approx \lambda^6$

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✗ Drawback: Barrington's transformation is very inefficient.

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- Key idea: embed additive group $(\mathbb{Z}_q, +)$ into a small symmetric group.

Overview of Bootstrapping Algorithm [AP'14]

- Decryption in LWE-based schemes is a "rounded inner product:"

$$\mathsf{Dec}(\mathbf{s}, \mathbf{c}) := \lfloor \langle \mathbf{s}, \mathbf{c} \rangle \rceil_2 \in \{0,1\} \text{ with } \mathbf{s} \in \mathbb{Z}_q^n,\ \mathbf{c} \in \{0,1\}^n$$

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We need two homomorphic algorithms for $\mathbb{Z}_q \subseteq G$:

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Overview of Bootstrapping Algorithm [AP'14]

- Decryption in LWE-based schemes is a "rounded inner product:"

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- It remains to define the group G and $\boxplus$, $\mathsf{Equal?}$ operations

Warmup: Embedding $(\mathbb{Z}_q, +)$ into $G = (S_q, \cdot)$

$$
\begin{array}{ccccc}
\mathbb{Z}_q & 0 & 1 & \dots & q-1 \\
S_q &
\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} &
\begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix} &
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- FFTW for fast ring operations $\Longrightarrow$ bootstrapping in 0.6 sec: FHEW!

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