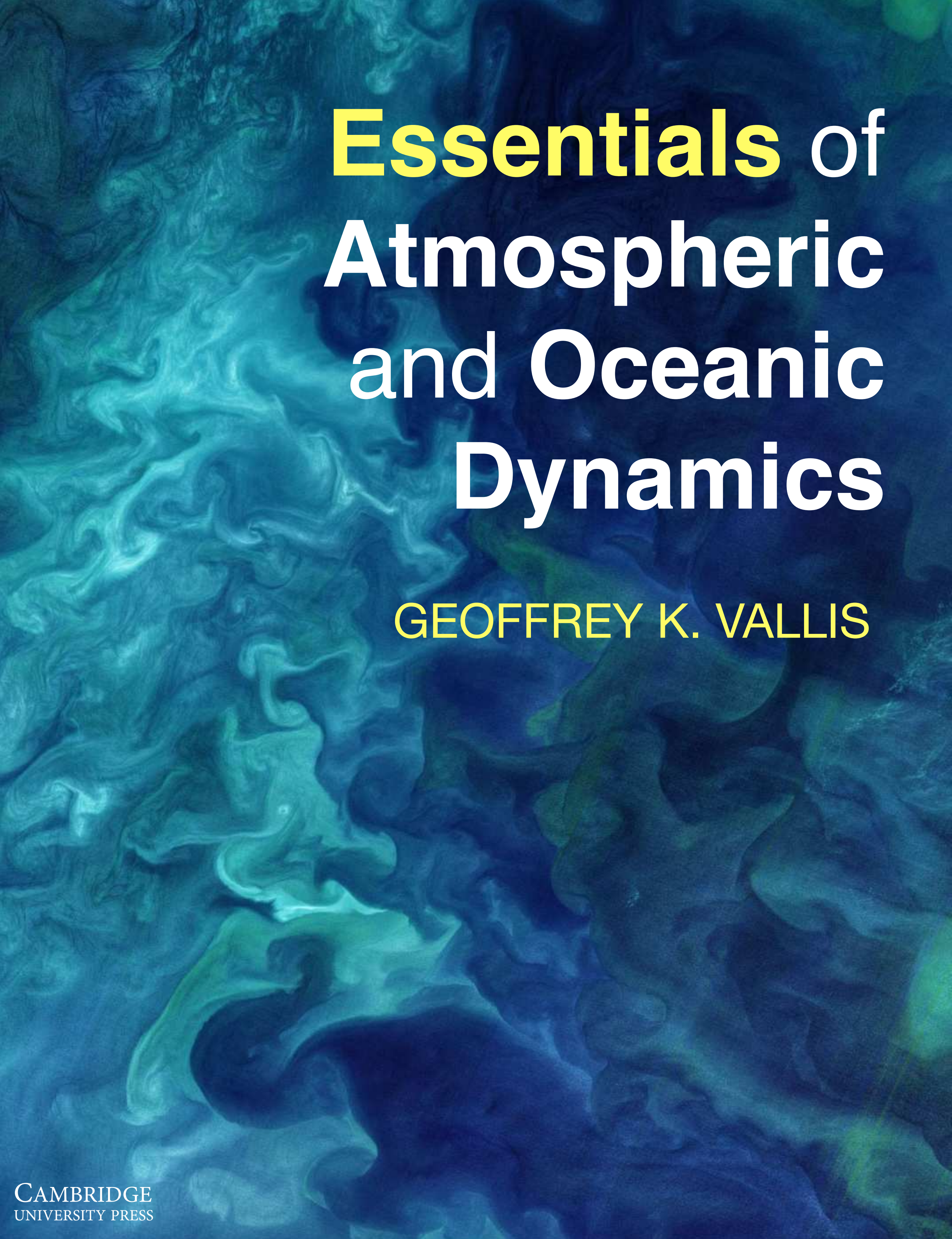

Essentials of Atmospheric and Oceanic Dynamics

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Contents

Chapter

$9\frac{3}{4}$

Circulation, Vorticity and Other Miscellany

This is a short supplementary chapter to *Essentials* containing miscellaneous material on vorticity, potential vorticity and the circulation theorem written at about the same level as the original book. At the end of the chapter I also give an alternate derivation of the quasi-geostrophic equations. Other material (e.g., gradient wind balance) may be added in due course — this is a living document. There is some repetition of material in order to avoid too much cross-referencing to the original book, but the chapter is not self-contained, nor are all derivations carried out in full detail. Occasionally we make reference to the Mother book, Atmospheric and Oceanic Fluid Dynamics, or AOFD, and that book has a more complete treatment of vorticity than will be found here. References to such things as (E6.17) refer to equations in *Essentials* whereas references such as (V.4) refer to equations in this chapter.

If you have any requests for material, or comments on this material, email me!

V.1 Vorticity and Circulation

V.1.1 Preliminaries

Vorticity, $\boldsymbol{\omega}$, is defined to be the curl of velocity and so is given by

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v}. \tag{V.1}$$

Circulation, C, is defined to be the integral of velocity around a closed fluid loop and so is given by

$$C \equiv \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}, \tag{V.2}$$

where the second expression uses Stokes' theorem and S is any surface bounded by the loop. The circulation around the path is equal to the integral of the normal component of vorticity over *any* surface bounded by that path. The circulation is not a field like vorticity and velocity; rather, we think of the circulation around a particular material line of finite length, and so its value generally depends on the path chosen. If δS is an infinitesimal surface element whose normal points in the direction of the unit vector $\hat{\boldsymbol{n}}$, then

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{v}) = \frac{1}{\delta S} \oint_{\delta r} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}, \tag{V.3}$$

where the line integral is around the infinitesimal area. Thus at a point the component of vorticity in the direction of $\boldsymbol{n}$ is proportional to the circulation around the surrounding infinitesimal fluid element, divided by the elemental area bounded by the path of the integral. A simple test for the presence of vorticity is to imagine a small paddle wheel in the flow; the paddle wheel acts as a 'circulation-meter', and rotates if the vorticity is non-zero. Vorticity might seem to be similar to angular momentum, in that it is a measure of spin. However, unlike angular momentum, *the value of vorticity at a point does not depend on the particular choice of an axis of rotation;* indeed, the definition of vorticity makes no reference at all to an axis of rotation or to a coordinate system. Rather, vorticity is a measure of the *local* spin of a fluid element.

V.1.2 Vorticity Equation

We may derive an evolution equation for vorticity by taking the curl of the momentum equation which, for a compressible fluid in a non-rotating frame of reference, is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{F} \tag{V.4}$$

where $\boldsymbol{F}$ represents all forcing and dissipation terms. Using the vector identity $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \nabla(\boldsymbol{v} \cdot \boldsymbol{v})/2 - (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$, we write the above equation as

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\omega} \times \boldsymbol{v} = -\frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \boldsymbol{v}^2 + \boldsymbol{F}, \tag{V.5}$$

Taking the curl of (V.5) gives the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = \frac{1}{\rho^2}(\nabla \rho \times \nabla p) + \nabla \times \boldsymbol{F}. \tag{V.6}$$

With some manipulation of vector identities (see AOFD for details) we can write this equation as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v} - \boldsymbol{\omega}\nabla \cdot \boldsymbol{v} + \frac{1}{\rho^2}(\nabla \rho \times \nabla p) + \nabla \times \boldsymbol{F}. \tag{V.7}$$

or as

$$\frac{\mathrm{D}\tilde{\boldsymbol{\omega}}}{\mathrm{D}t} = (\tilde{\boldsymbol{\omega}} \cdot \nabla)\boldsymbol{v} + \frac{1}{\rho^3}(\nabla \rho \times \nabla p) + \frac{1}{\rho}\nabla \times \boldsymbol{F}, \tag{V.8}$$

where $\tilde{\boldsymbol{\omega}} \equiv \boldsymbol{\omega}/\rho$.

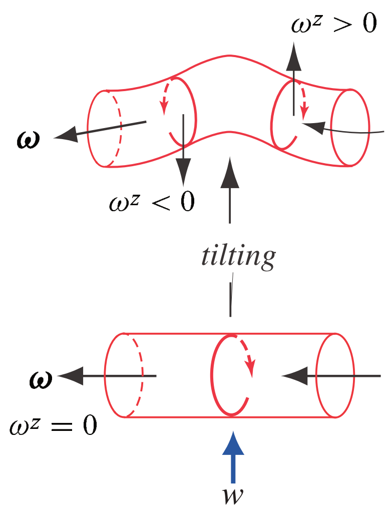

Fig. V.1: The tilting of vorticity. Suppose that the vorticity, $\boldsymbol{\omega}$ is initially directed horizontally, as in the lower figure, so that ω^z, its vertical component, is zero. The material lines, and therefore also the vortex lines, are tilted by the positive vertical velocity w thus creating a non-zero vertically oriented vorticity. This mechanism is important in creating vertical vorticity in the atmospheric boundary layer, and is connected to the β-effect in large-scale flow.

Terms in the Equation

The third term on the right-hand side of (V.7), as well as the second term on the right-hand side of (V.8), is variously called the *baroclinic* term, the *non-homentropic* term, or the *solenoidal* term. The solenoidal vector, $\boldsymbol{S}_o$, is defined by

$$\boldsymbol{S}_o \equiv \frac{1}{\rho^2}\nabla\rho \times \nabla p = -\nabla\alpha \times \nabla p. \tag{V.9}$$

where $\alpha = 1/\rho$. A solenoid is a tube directed perpendicular to both $\nabla\alpha$ and ∇p, with elements of length proportional to $\nabla p \times \nabla\alpha$. If the isolines of p and α are parallel to each other, then solenoids do not exist. *This occurs when the density is a function only of pressure,* for then

$$\nabla\rho \times \nabla p = \nabla\rho \times \nabla\rho \frac{\mathrm{d}p}{\mathrm{d}\rho} = 0. \tag{V.10}$$

Using the fundamental thermodynamic relation (see AOFD for the derivation), the solenoidal vector may also be written as

$$\boldsymbol{S}_o = -\nabla\eta \times \nabla T. \tag{V.11}$$

where η is the entropy and T the temperature. (This form can be useful when deriving potential vorticity conservation, which we come to later.) Evidently the solenoidal term vanishes if: (i) isolines of pressure and density are parallel; (ii) isolines of temperature and entropy are parallel; (iii) density, entropy, temperature or pressure are constant. A *barotropic* fluid has by definition $\rho = \rho(p)$ and therefore no solenoids. A *baroclinic* fluid is one for which ∇p is not parallel to $\nabla\rho$. From (V.8) we see that the baroclinic term must be balanced by terms involving velocity or its tendency and therefore, in general, *a baroclinic fluid is a moving fluid,* even in the presence of viscosity.

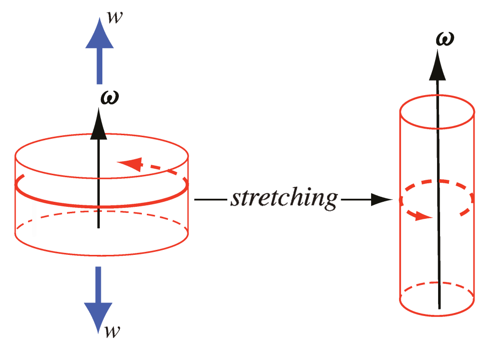

Fig. V.2: A vertical velocity, w, stretches the cylinder. Vorticity is tied to material lines and so is amplified in the direction of the stretching. However, because the volume of fluid is conserved, the end surfaces shrink, the material lines through the cylinder ends converge and the integral of vorticity over a material surface (the circulation) remains constant.

For a barotropic fluid (and therefore with no solenoidal term) the vorticity equation takes the simple form,

$$\frac{\mathrm{D}\tilde{\boldsymbol{\omega}}}{\mathrm{D}t} = (\tilde{\boldsymbol{\omega}} \cdot \nabla)\boldsymbol{v}. \tag{V.12}$$

If the fluid is also incompressible, meaning that $\nabla \cdot \boldsymbol{v} = 0$, then we have the even simpler form,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{v}. \tag{V.13}$$

When expanded into components, the terms on the right-hand side of (V.12) or (V.13) can be divided into 'stretching' and 'tipping' (or 'tilting'). These terms can be illustrated if we consider one component, for example the vertical component, of (V.13), namely

$$\begin{aligned} \frac{\mathrm{D}\omega^z}{\mathrm{D}t} &= (\boldsymbol{\omega} \cdot \nabla)w \\ &= \omega^x \frac{\partial w}{\partial x} + \omega^y \frac{\partial w}{\partial y} + \omega^z \frac{\partial w}{\partial z} \end{aligned} \tag{V.14}$$

The first two terms on the right-hand side give rise to the tilting of vorticity, with one component of vorticity essentially being converted to another, as illustrated in Fig. V.1. The stretching of vorticity arises from the last term on the right-hand side, namely

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = \zeta \frac{\partial w}{\partial z}, \tag{V.15}$$

where, as is conventional, ζ is the vertical component of the vorticity, ω^z and we omit the other terms. If the vertical velocity w increases in the z-direction then the fluid lines are 'stretched' and the vorticity is amplified, as illustrated in Fig. V.2.

V.1.3 Vorticity in Two-dimensional Flow

Let us suppose that there is flow only in a single plane, the x–y plane, with no flow in the normal direction (the z-direction). The fluid velocity in the plane, $\boldsymbol{u}$, is $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$, and the velocity normal to the plane, w, is zero. Only one component of vorticity is non-zero and this is

$$\boldsymbol{\omega} = \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{V.16}$$

That is, in two-dimensional flow the vorticity is perpendicular to the velocity. We let $\zeta \equiv \omega^z = \boldsymbol{\omega}\cdot\mathbf{k}$. Both the stretching and tilting terms vanish in two-dimensional flow, and the two-dimensional vorticity equation becomes, for incompressible flow,

$$\frac{\mathrm{D}\zeta}{\mathrm{D}t} = 0, \tag{V.17}$$

where $\mathrm{D}\zeta/\mathrm{D}t = \partial\zeta/\partial t + \boldsymbol{u}\cdot\nabla\zeta$. That is, *in inviscid two-dimensional flow vorticity is conserved following the fluid elements*; each material parcel of fluid keeps its value of vorticity even as it is being advected around. Furthermore, specification of the vorticity completely determines the flow field. To see this, we use the incompressibility condition to define a streamfunction ψ such that

$$u = -\frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x}, \qquad \zeta = \nabla^2\psi. \tag{1.18a,b,c}$$

Given the vorticity, the Poisson equation (V.18c) can be solved for the streamfunction and the velocity fields obtained through (V.18a,b), and this process is called 'inverting the vorticity'.

V.2 VORTICITY EQUATION IN A ROTATING FRAME

V.2.1 The General Form

Let us denote the velocity and vorticity in the absolute frame by $\boldsymbol{\omega}_a$ and $\boldsymbol{v}_a$, and in the rotating frame by $\boldsymbol{\omega}_r$ and $\boldsymbol{v}_r$. The quantities are related by

$$\boldsymbol{\omega}_a = \boldsymbol{\omega}_r + 2\boldsymbol{\Omega}, \qquad \boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r}, \tag{V.19}$$

where $\boldsymbol{r}$ is the position vector relative to the axis of rotation. (The absolute velocity is also sometimes referred to as the inertial velocity.) Now, the momentum equation in the rotating frame of reference may be written as

$$\frac{\partial\boldsymbol{v}_r}{\partial t} + (2\boldsymbol{\Omega} + \boldsymbol{\omega}_r)\times\boldsymbol{v}_r = -\frac{1}{\rho}\nabla p - \nabla\left(\Phi + \frac{1}{2}\boldsymbol{v}_r^2\right), \tag{V.20}$$

where potential Φ contains the gravitational ($\boldsymbol{g}$) and centrifugal (forces and we neglect viscosity. Take the curl of this equation, using the vector identity

$$\nabla\times[(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\times\boldsymbol{v}_r] = (2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + (\boldsymbol{v}_r\cdot\nabla)(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r) - [(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\cdot\nabla]\boldsymbol{v}_r, \tag{V.21}$$

and noting that $\nabla\cdot(2\boldsymbol{\Omega}+\boldsymbol{\omega}) = 0$, we obtain the vorticity equation

$$\frac{\mathrm{D}\boldsymbol{\omega}_r}{\mathrm{D}t} = [(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\cdot\nabla]\,\boldsymbol{v} - (2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\nabla\cdot\boldsymbol{v}_r + \frac{1}{\rho^2}(\nabla\rho\times\nabla p). \tag{V.22}$$

If the rotation rate, $\boldsymbol{\Omega}$, is a constant then $\mathrm{D}\boldsymbol{\omega}_r/\mathrm{D}t = \mathrm{D}\boldsymbol{\omega}_a/\mathrm{D}t$ where $\boldsymbol{\omega}_a = 2\boldsymbol{\Omega}+\boldsymbol{\omega}_r$ is the absolute vorticity. The only difference between the vorticity equation in the rotating and inertial frames of reference is in the presence of the solid-body vorticity $2\boldsymbol{\Omega}$ on the right-hand side. The second term on the right-hand side may be folded into the material derivative using mass continuity, and after a little manipulation (V.22) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a}{\rho}\right) = \frac{1}{\rho}(2\boldsymbol{\Omega}+\boldsymbol{\omega}_r)\cdot\nabla\boldsymbol{v}_r + \frac{1}{\rho^3}(\nabla\rho\times\nabla p). \tag{V.23}$$

Note that it is the *absolute* vorticity, $\boldsymbol{\omega}_a$, that now appears on the left-hand side. If ρ is constant, $\boldsymbol{\omega}_a$ may be replaced by $\boldsymbol{\omega}_r$.

V.2.2 The Vertical Component of the Vorticity Equation

In large-scale dynamics, the most important, although not the largest, component of the vorticity is often the vertical one, because this contains much of the information about the horizontal flow. We can obtain an explicit expression for its evolution by taking the vertical component of (V.22), but this requires some care. An alternative derivation begins with the horizontal momentum equations,

$$\frac{\partial u}{\partial t} - v(\zeta+f) + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2) \tag{V.24a}$$

$$\frac{\partial v}{\partial t} + u(\zeta+f) + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{2}\frac{\partial}{\partial y}(u^2+v^2), \tag{V.24b}$$

or, equivalently,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{V.25a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{V.25b}$$

where we no drop the subscript r on variables measured in the rotating frame. Cross-differentiating either of the above pairs of equations gives, after a little algebra,

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\zeta+f) = -(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right)\\ + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right).\end{aligned} \tag{V.26}$$

We interpret the various terms as follows:

$\mathrm{D}\zeta/\mathrm{D}t = \partial\zeta/\partial t + \boldsymbol{v}\cdot\nabla\zeta$. The material derivative of the vertical component of the vorticity.

$\mathrm{D}f/\mathrm{D}t = v\partial f/\partial y = v\beta$. The β-effect. The vorticity is affected by the meridional motion of the fluid, so that, apart from the terms on the right-hand side, $(\zeta+f)$ is conserved on parcels. Because the Coriolis parameter changes with latitude this is like saying that the system has differential rotation.

$-(\zeta + f)(\partial u/\partial x + \partial v/\partial y)$. The divergence term, which gives rise to vortex stretching. In an incompressible fluid this may be written $(\zeta + f)\partial w/\partial z$, so that vorticity is amplified if the vertical velocity increases with height, so stretching the material lines and the vorticity.

$(\partial u/\partial z)(\partial w/\partial y) - (\partial v/\partial z)(\partial w/\partial x)$. The tilting term, whereby a vertical component of vorticity may be generated by a vertical velocity acting on a horizontal vorticity.

$\rho^{-2}\left[(\partial\rho/\partial x)(\partial p/\partial y) - (\partial\rho/\partial y)(\partial p/\partial x)\right] = \rho^{-2}J(\rho, p)$. The solenoidal term arising when isosurfaces of pressure and density are not parallel.

In large-scale atmospheric and oceanic applications the second and third terms on the right-hand side of (V.26) are normally smaller than the other terms and the equation becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{V.27}$$

This is the starting point for many additional approximations commonly made in geophysical fluid dynamics. The most severe approximation is to assume there is no vertical motion at all, as we now see.

Two-dimensional and shallow water vorticity equations

In an inviscid two-dimensional incompressible flow, all of the terms on the right-hand side of (V.26) vanish and we have the simple equation

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} = 0, \tag{V.28}$$

implying that the absolute vorticity, $\zeta_a \equiv \zeta + f$, is materially conserved. If f is a constant then $\mathrm{D}f/\mathrm{D}t = 0$ and background rotation plays no role. If f varies linearly with y, so that $f = f_0 + \beta y$, then (V.28) becomes

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{u}\cdot\nabla\zeta + \beta v = 0, \tag{V.29}$$

which is known as the two-dimensional β-plane vorticity equation.

For inviscid shallow water flow the horizontal divergence is non-zero and we easily find. can show that (see also Chapter 4)

$$\frac{\mathrm{D}(\zeta + f)}{\mathrm{D}t} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{V.30}$$

V.3 Kelvin's Circulation Theorem

Kelvin's circulation theorem states that under certain circumstances the circulation around a material fluid parcel is conserved; that is, the circulation is conserved 'moving with the flow'. The primary restrictions are that body forces are conservative (i.e., they are representable as potential force), that the flow be inviscid, and that the fluid is barotropic with $\rho = \rho(p)$. Of these, the latter is the most restrictive for geophysical fluids (note that gravity can be written as a potential $\boldsymbol{g} = -\nabla\phi$ where $\phi = -gz$). A constant density, inviscid flow satisfies these conditions.

To prove the theorem, we begin with the inviscid momentum equation,

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p - \nabla\Phi, \tag{V.31}$$

where $\nabla\Phi$ represents the conservative body forces on the system. Applying the material derivative to the circulation, (V.2), gives

$$\begin{aligned} \frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r} &= \oint\left(\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\cdot\mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{v}\right) \\ &= \oint\left[\left(-\frac{1}{\rho}\nabla p - \nabla\Phi\right)\cdot\mathrm{d}\boldsymbol{r} + \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{v}\right] \\ &= \oint -\frac{1}{\rho}\nabla p\cdot\mathrm{d}\boldsymbol{r}, \end{aligned} \tag{V.32}$$

using (V.31) and $D(\mathrm{d}\boldsymbol{r})/Dt = \mathrm{d}\boldsymbol{v}$, where $\mathrm{d}\boldsymbol{r}$ is the line element and with the line integration being over a closed, material, circuit. The second and third terms on the second line vanish separately, because they are exact differentials integrated around a closed loop. The term on the last line vanishes if the density is constant or, more generally, if the density is a function of pressure alone (see section (4.3.2) in AOFD to prove that). Equation (V.32) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r} = 0. \tag{V.33}$$

This is Kelvin's circulation theorem. In words, *the circulation around a material loop is invariant for a barotropic fluid that is subject only to conservative forces.* Using Stokes' theorem, the circulation theorem may also be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_S \boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S} = 0. \tag{V.34}$$

That is, the area integral of the normal component of vorticity across any material surface is constant, under the same conditions. This form is both natural and useful, and it arises because of the way vorticity is tied to material fluid elements. Kelvin's circulation theorem is a conservation law that is unique to fluids. Unlike, say, the conservation of energy, it has no analogue in solid body mechanics. Potential vorticity conservation, which we come to later on, is an extension of circulation conservation.

Stretching and circulation

Let us informally consider how vortex stretching and mass conservation work together to give the circulation theorem. Let the fluid be incompressible so that the volume of a fluid mass is constant, and consider a surface normal to a vortex tube, as in Fig. V.2. Let the volume of a small material box around the surface be δV, the length of the material lines be δl and the surface area be δA. Then

$$\delta V = \delta l \, \delta A. \tag{V.35}$$

Because of the frozen-in property, the vorticity passing through the surface is proportional to the length of the material lines. That is, $\omega \propto \delta l$, and

$$\delta V \propto \omega \, \delta A. \tag{V.36}$$

The right-hand side is just the circulation around the surface. Now, if the corresponding material tube is stretched δl increases, but the volume, δV, remains constant by mass conservation. Thus, the circulation given by the right-hand side of (V.36) also remains constant. In other words, because of the frozen-in property vorticity is amplified by the stretching, but the vortex lines get closer together in such a way that the product $\omega \, \delta A$ remains constant and circulation is conserved.

Baroclinic Circulation

In baroclinic flow, the circulation is not generally conserved, and from (V.32) we have

$$\frac{\mathrm{D}C}{\mathrm{D}t} = -\oint \frac{\nabla p}{\rho} \cdot \mathrm{d}\boldsymbol{r} = -\oint \frac{\mathrm{d}p}{\rho}, \tag{V.37}$$

and this is sometimes called the baroclinic circulation theorem.

There are a some manipulations of this that will be useful when considering potential vorticity. The fundamental thermodynamic relation states that $T\,\mathrm{d}\eta = \mathrm{d}I + p\,\mathrm{d}\alpha$, where η is entropy and $\alpha = \rho^{-1}$. We therefore have

$$\alpha \, \mathrm{d}p = \mathrm{d}(p\alpha) - T\,\mathrm{d}\eta + \mathrm{d}I, \tag{V.38}$$

and the first and last terms on the right-hand side will vanish upon integration around a circuit. The solenoidal term on the right-hand side of (V.37) may therefore be written as

$$S_o \equiv -\oint \alpha \, \mathrm{d}p = \oint T\,\mathrm{d}\eta = -\oint \eta \, \mathrm{d}T = -R\oint T\,\mathrm{d}\log p, \tag{V.39}$$

where the last equality holds only for an ideal gas. Using Stokes' theorem to the first equality in (V.37), S_o can also be written as

$$S_o = -\int_S (\nabla\alpha \times \nabla p) \cdot \mathrm{d}\boldsymbol{S} = -\int_S \left(\frac{\partial\alpha}{\partial T}\right)_p (\nabla T \times \nabla p) \cdot \mathrm{d}\boldsymbol{S} = \int_S (\nabla T \times \nabla \eta) \cdot \mathrm{d}\boldsymbol{S}. \tag{V.40}$$

The rate of change of the circulation across a surface depends on the existence of this solenoidal term (Fig. V.3).

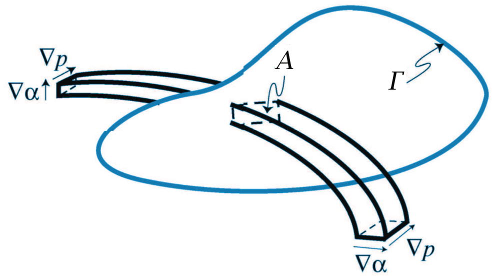

Fig. V.3: Solenoids and the circulation theorem. Solenoids are tubes perpendicular to both $\nabla\alpha$ and ∇p, and they have a non-zero cross-sectional area if isolines of α and p do not coincide. The rate of change of circulation over a material surface is given by the sum of all the solenoidal areas crossing the surface. If $\nabla\alpha \times \nabla p = 0$ there are no solenoids.

However, even if the solenoidal vector is in non-zero, circulation *is* conserved if the material path is in a surface of constant entropy, η, *and if* $\mathrm{D}\eta/\mathrm{D}t = 0$. The solenoidal term then vanishes and, because $\mathrm{D}\eta/\mathrm{D}t = 0$, entropy remains constant on that same material loop as it evolves. This result gives rise to the conservation of potential vorticity, discussed in Section V.4.

V.3.1 Circulation in a Rotating Frame

The absolute and relative velocities are related by $\boldsymbol{v}_a = \boldsymbol{v}_r + \boldsymbol{\Omega} \times \boldsymbol{r}$, so that in a rotating frame the rate of change of circulation is given by

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} = \oint\left[\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + \boldsymbol{\Omega}\times\boldsymbol{v}_r\right)\cdot\mathrm{d}\boldsymbol{r} + (\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{v}_r\right]. \tag{V.41}$$

But $\oint \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{v}_r = 0$ and, integrating by parts,

$$\begin{aligned}\oint(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{v}_r &= \oint\left\{\mathrm{d}[(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\boldsymbol{v}_r] - (\boldsymbol{\Omega}\times\mathrm{d}\boldsymbol{r})\cdot\boldsymbol{v}_r\right\} \\ &= \oint\left\{\mathrm{d}[(\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\boldsymbol{v}_r] + (\boldsymbol{\Omega}\times\boldsymbol{v}_r)\cdot\mathrm{d}\boldsymbol{r}\right\}.\end{aligned} \tag{V.42}$$

The first term on the right-hand side is zero and so (V.41) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} = \oint\left(\frac{\mathrm{D}\boldsymbol{v}_r}{\mathrm{D}t} + 2\boldsymbol{\Omega}\times\boldsymbol{v}_r\right)\cdot\mathrm{d}\boldsymbol{r} = -\oint\frac{\mathrm{d}p}{\rho}, \tag{V.43}$$

where the second equality uses the momentum equation. The last term vanishes if the fluid is barotropic, and if so the circulation theorem is

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint(\boldsymbol{v}_r+\boldsymbol{\Omega}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} = 0, \qquad \text{or} \qquad \frac{\mathrm{D}}{\mathrm{D}t}\int_S(\boldsymbol{\omega}_r + 2\boldsymbol{\Omega})\cdot\mathrm{d}\boldsymbol{S} = 0, \tag{V.44a,b}$$

where the second equation uses Stokes' theorem and we have used $\nabla\times(\boldsymbol{\Omega}\times\boldsymbol{r}) = 2\boldsymbol{\Omega}$, and where $\boldsymbol{\omega}_r = \nabla\times\boldsymbol{v}_r$ is the relative vorticity. These results could have easily been guessed: the term $\boldsymbol{v}_r + \boldsymbol{\Omega}\times\boldsymbol{r}$ is just the absolute velocity, and the term $\boldsymbol{\omega}_r + 2\boldsymbol{\Omega}$ is just the absolute vorticity. That is, the results in the rotating frame just use the absolute velocity and vorticity.

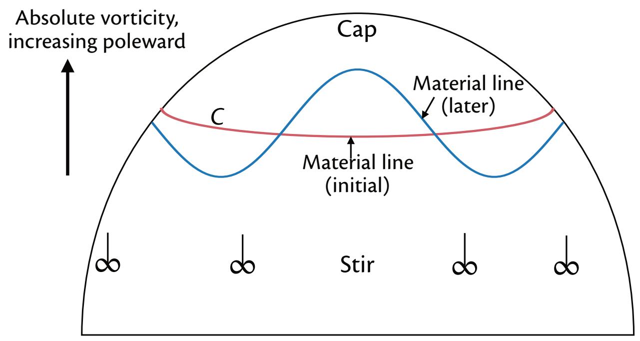

Fig. V.4: Sketch of the effects of a midlatitude disturbance on the circulation around the latitude line C. If initially the absolute vorticity increases monotonically poleward, then the disturbance will bring fluid with lower absolute vorticity into the cap region. Then, using Stokes theorem, the velocity around the latitude line C will become more westward.

V.3.2 An application to Earth's Atmosphere

W now consider an example of circulation relevant to Eart's atmosphere, one that we come back to in Chapter 12 of *Essentials*. We suppose that the Earth's atmosphere is two dimensional (latitude and longitude) and that absolute vorticity normal to the surface , $\zeta + f$, where $f = 2\Omega \sin\vartheta$, increases monotonically poleward. By Stokes' theorem, the initial circulation, I_i, around a line of latitude circumscribing the polar cap (the red line in Fig. V.4) is equal to the integral of the absolute vorticity over the cap. That is,

$$I_i = \int_{\text{cap}} \boldsymbol{\omega}_{ia} \cdot \mathrm{d}\boldsymbol{A} = \oint_C u_{ia}\, \mathrm{d}l = \oint_C (u_i + \Omega a \cos\vartheta)\, \mathrm{d}l, \tag{V.45}$$

where $\boldsymbol{\omega}_{ia}$ and u_{ia} are the initial absolute vorticity and absolute velocity, respectively, u_i is the initial zonal velocity in the Earth's frame of reference, and the line integrals are around the line of latitude. Suppose the fluid is initally at rest (so that $u_i = 0$) and there is a disturbance equatorward of the polar cap, and that this results in a distortion of the material line around the latitude circle C (Fig. 12.1).

Since the source of the disturbance is distant from the latitude of interest, if we neglect viscosity the circulation along the material line is conserved, by Kelvin's circulation theorem, so that the circulation around the blue line is equal to the circulation around the red line. Now, vorticity with a lower value is brought into the region of the polar cap — that is, the region poleward of the latitude line C. Using Stokes' theorem again the circulation around the latitude circle C must therefore fall; that is, denoting later values with a subscript f,

$$I_f = \int_{\text{cap}} \boldsymbol{\omega}_{fa} \cdot \mathrm{d}\boldsymbol{A} < I_i\,, \tag{V.46}$$

so that

$$\oint_C (u_f + \Omega a \cos\vartheta)\, \mathrm{d}l < \oint_C (u_i + \Omega a \cos\vartheta)\, \mathrm{d}l, \tag{V.47}$$

and thus

$$\overline{u}_f < \overline{u}_i, \tag{V.48}$$

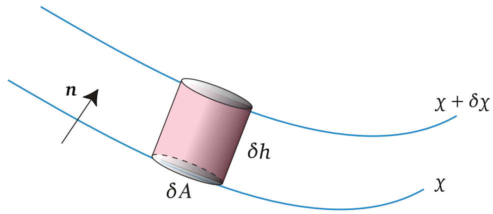

Fig. V.5: An infinitesimal fluid element, bounded by two isosurfaces of the conserved tracer χ. As $\mathrm{D}\chi/\mathrm{D}t = 0$, then $\mathrm{D}\delta\chi/\mathrm{D}t = 0$.

with the overbar indicating a zonal average. Thus, there is a tendency to produce *westward* flow poleward of the disturbance. This result has important ramifications for the production of eastward (i.e., or westerly) winds in the mid-latitude atmosphere.

V.4 Potential Vorticity Conservation

Although Kelvin's circulation theorem is a general statement about vorticity conservation, in its original form it is not always a practically useful statement for two reasons. First, it is not a statement about a *field*, such as vorticity itself. Second, it is not satisfied for baroclinic flow, such as is found in the atmosphere and ocean. It turns out that it is possible to derive a beautiful conservation law that overcomes both of these failings and one that, furthermore, is extraordinarily useful in geophysical fluid dynamics. This is the conservation of *potential vorticity* (PV) introduced first by Rossby and then in a more general form by Ertel.

V.4.1 PV Conservation from the Circulation Theorem

Barotropic fluids

Let us begin with the simple case of a barotropic fluid. For an infinitesimal fluid element we write Kelvin's theorem as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\,\delta A\right] = 0, \tag{V.49}$$

where $\boldsymbol{n}$ is a unit vector normal to an infinitesimal surface δA. This is just a form of (V.34) applied over a very small surface. Now consider a volume bounded by two isosurfaces of values χ and $\chi + \delta\chi$, where χ is any materially conserved tracer, thus satisfying $\mathrm{D}\chi/\mathrm{D}t = 0$, so that δA initially lies in an isosurface of χ (see Fig. V.5). Since $\boldsymbol{n} = \nabla\chi/|\nabla\chi|$ and the infinitesimal volume $\delta V = \delta h\,\delta A$, where δh is the separation between the two surfaces, we have

$$\boldsymbol{\omega}_a \cdot \boldsymbol{n}\,\delta A = \boldsymbol{\omega}_a \cdot \frac{\nabla\chi}{|\nabla\chi|}\frac{\delta V}{\delta h}. \tag{V.50}$$

Now, the value of δh may be obtained from

$$\delta\chi = \delta\boldsymbol{x} \cdot \nabla\chi = \delta h|\nabla\chi|, \tag{V.51}$$

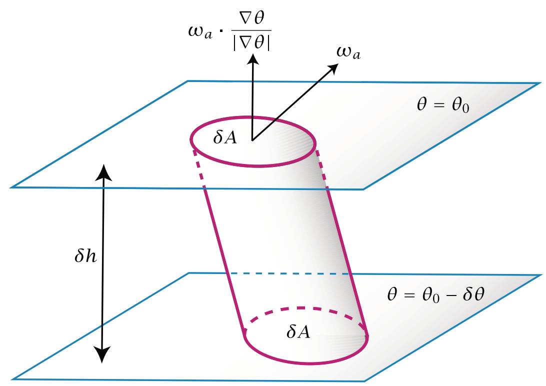

Fig. V.6: Geometry of potential vorticity conservation. The circulation equation is $\mathrm{D}[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A]/\mathrm{D}t = \boldsymbol{S}_o \cdot \boldsymbol{n}\,\delta A$, where $\boldsymbol{S}_o \propto \nabla\theta \times \nabla T$. We choose $\boldsymbol{n} = \nabla\theta/|\nabla\theta|$, where θ is materially conserved, to annihilate the solenoidal term on the right-hand side, and we note that $\delta A = \delta V/\delta h$, where δV is the volume of the cylinder, and the height of the column is $\delta h = \delta\theta/|\nabla\theta|$. The circulation is $C \equiv \boldsymbol{\omega}_a \cdot \boldsymbol{n}\,\delta A = \boldsymbol{\omega}_a \cdot (\nabla\theta/|\nabla\theta|)(\delta V/\delta h) = [\rho^{-1}\boldsymbol{\omega}_a \cdot \nabla\theta](\delta M/\delta\theta)$, where $\delta M = \rho\,\delta V$ is the mass of the cylinder. As δM and $\delta\theta$ are materially conserved, so is the potential vorticity $\rho^{-1}\boldsymbol{\omega}_a \cdot \nabla\theta$.

and using this expression and (V.50) in (V.49) we obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\left[\frac{(\boldsymbol{\omega}_a \cdot \nabla\chi)\delta V}{\delta\chi}\right] = 0. \tag{V.52}$$

Now, we are presuming that χ is conserved on material elements, and therefore so is $\delta\chi$ and it may be taken out of the differentiation. The mass of the volume element $\rho\,\delta V$ is also conserved, so that (V.52) becomes

$$\frac{\rho\delta V}{\delta\chi}\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a}{\rho} \cdot \nabla\chi\right) = 0 \tag{V.53}$$

or

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega} \cdot \nabla\chi}{\rho}\right) = 0, \tag{V.54}$$

Equation (V.54) is a statement of potential vorticity conservation for a barotropic fluid. The field χ may be chosen arbitrarily, provided that it is materially conserved.

The baroclinic case

For a baroclinic fluid the above derivation fails simply because the statement of the conservation of circulation, (V.49) is not, in general, true: there are solenoidal terms on the right-hand side and from (V.37) and (V.40) we have

$$\frac{\mathrm{D}}{\mathrm{D}t}[(\boldsymbol{\omega}_a \cdot \boldsymbol{n})\delta A] = \boldsymbol{S}_o \cdot \boldsymbol{n}\,\delta A, \qquad \boldsymbol{S}_o = -\nabla\alpha \times \nabla p = -\nabla\eta \times \nabla T. \tag{V.55a,b}$$

However, the right-hand side of (V.55a) may be annihilated by choosing the circuit around which we evaluate the circulation to be such that the solenoidal term is identically zero. Given the form of $\boldsymbol{S}_o$, this occurs if the values of any of p, ρ, η, T are constant on that circuit; that is, if $\chi = p, \rho, \eta$ or T. But the derivation also demands that χ be a materially conserved quantity, which usually restricts the choice of χ to be η (or, more usually for an ideal gas, potential temperature, θ, since $\eta = c_p \ln\theta$). In the ocean we often take $\chi = b$, the buoyancy. Thus, the conservation of potential vorticity for inviscid, adiabatic flow in an ideal gas is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_a \cdot \nabla\theta}{\rho}\right) = 0, \tag{V.56}$$

where $\mathrm{D}\theta/\mathrm{D}t = 0$. (For diabatic and viscous flow various source terms appear on the right-hand side.) A summary of this derivation is provided by Fig. V.6.

V.5 Alternative Derivations of Quasi-Geostrophy

In this section we provide a couple of other ways to derive the quasi-geostrophic equations, for both the shallow-water equations and the continuously-stratified Boussinesq equations. The first method begins with the potential vorticity equation and makes simplifications to it; this method is revealing of the true essence of quasi-geotrophy. The second method, which is perhaps more straightforward, begins with the momentum equations and makes approximations to it.

V.5.1 Quasi-Geostrophy from Potential Vorticity

The evolution equation in quasi-geostrophic theory is an approximation to the evolution of potential vorticity. The correspondence is straightforward in the shallow water case but less so in the continuously stratified case – see AOFD for some important caveats. Our derivation will be most informative for those who have read a more conventional derivation in *Essentials* or elsewhere.

Shallow water equations

This method is already described, albeit briefly, in section 5.2 of Essentials. The potential vorticity equation is

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0, \tag{V.57}$$

where

$$Q = \frac{f + \zeta}{h}, \tag{V.58}$$

where h is the fluid thickness. We set bottom topography to zero so that h is also the height of the free surface, and then let $h = H(1 + \eta_T/H)$ where H is the mean thickness and η_T is the deviation height above this level. We will assume that η_T/H is small, and, if $f = f_0 + \beta y$ and $|\beta y| \ll f_0$, we obtain

$$Q = \frac{f + \zeta}{H(1 + \eta_T/H)} \tag{V.59}$$

$$\approx \frac{1}{H}(f + \zeta)\left(1 - \frac{\eta_T}{H}\right) \qquad \text{since } \eta_t \text{ is small,} \tag{V.60}$$

$$\approx \frac{1}{H}\left(f_0 + \beta y + \zeta - f_0\frac{\eta_T}{H}\right) \qquad \text{since } \beta y \text{ is small.} \tag{V.61}$$

Because f_0/H is a constant it has no effect in the evolution equation, and the quantity given by

$$q = \beta y + \zeta - f_0\frac{\eta_T}{H} \tag{V.62}$$

is materially conserved. This is the quasi-geostrophic potential vorticity. We still have two unknowns, ζ and η_T, but these are related by geostrophic balance. Since $|f_0| \gg |\beta y|$ we have

$$f_0 u = -g\frac{\partial \eta_T}{\partial y}, \quad f_0 v = g\frac{\partial \eta_T}{\partial x} \qquad \text{or} \qquad u = -\frac{\partial \psi}{\partial y}, \quad v = g\frac{\partial \psi}{\partial x} \tag{V.63}$$

where $\psi = g\eta_T/f_0$. The vorticity is thus given by $\zeta = \nabla^2\psi$ and (V.62) may be written as

$$q = \beta y + \nabla^2\psi - \frac{\psi}{L_d^2}, \tag{V.64}$$

where $L_d = \sqrt{gH}/f_0$. The quasi-geostrophic evolution equation is then simply written in the equivalent forms

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0 \quad \text{or} \quad \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} = 0 \quad \text{or} \quad \frac{\partial q}{\partial t} + J\psi, q) = 0, \tag{V.65}$$

where the advecting velocity is the geostrophic one and $J(\psi, q) \equiv (\partial\psi/\partial x)(\partial q/\partial y) - (\partial\psi/\partial q)\,(\partial q/\partial x)$ is known as the *Jacobian* of ψ and q.

Stratified, Boussinesq equations

We will start with (V.56), but restrict attention to a Boussinesq fluid, for which ρ is a a constant. The PV evolution is then, in a rotating frame of reference,

$$\frac{\mathrm{D}Q}{\mathrm{D}t} = 0, \qquad Q = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla b, \tag{V.66}$$

where $b = g\,\delta\theta/\theta_0$ and θ_0 is a constant. We shall assume the fluid is in hydrostatic balance and expanding the derivatives we obtain

$$Q = (\boldsymbol{\omega} + \boldsymbol{f}) \cdot \nabla b \approx (\zeta + f)b_z - v_z b_x + u + z b_y. \tag{V.67}$$

where subscripts denote derivative and the last two terms arise from the horizontal vorticity, with terms involving vertical velocity being absent in the hydrostatic approximation. These last two terms are in fact usually small in the atmosphere and ocean and we will ignore them.

The variation of b, the buoyancy, is largest in the vertical direction, and $\boldsymbol{\Omega}$ also is directed in the vertical direction. We thus write

$$2\boldsymbol{\Omega} = f\mathbf{k} = (f_0 + \beta y)\mathbf{k} \tag{V.68}$$

and

$$b = b_0(z) + b'(x, y, z, t). \tag{V.69}$$

where b_0 is the dominant variability in the vertical and $N^2 = \mathrm{d}b_0/\mathrm{d}z$ is the static stability. The potential vorticity is then, approximately,

$$\begin{aligned} Q &\approx (\zeta + f_0 + \beta y)\left(N^2 + \frac{\partial b'}{\partial z}\right) \\ &\approx f_0\left(N^2 + \frac{\partial b'}{\partial z}\right) + N^2(\zeta + \beta y) \end{aligned} \tag{V.70}$$

where the approximation on the second line arises because $|f_0| \gg \zeta$ (the small Rossby number assumption) and $N^2 \gg \partial b'/\partial z$. We do not neglect $f_0 \partial b'/\partial z$ because f_0 is big. Now, $f_0 N^2$ is constant in time and in x and y, so it's horizontal advection is zero. Furthermore, since the flow is nearly in geostrophic balance the vertical velocity is very small and we therefore neglect it. The potential vorticity equation becomes

$$\frac{\partial Q}{\partial t} + \boldsymbol{u} \cdot \nabla Q = 0, \qquad Q = N^2(\zeta + \beta y) + f_0 \frac{\partial b' z}{\partial,} \tag{V.71}$$

which can be written as

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \nabla q = 0, \qquad q = (\zeta + \beta y) + \frac{f_0}{N^2}\frac{\partial b'}{\partial z}. \tag{V.72}$$

Equation (V.72) is almost, but not exactly, the same as the quasi-geostrophic potential vorticity equation (E5.61) given in *Essentials*, namely

$$\frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \nabla q = 0, \qquad q = \zeta + f + \frac{\partial}{\partial z}\left(\frac{f_0 b'}{N^2}\right). \tag{V.73}$$

The subtle differences in the last term arise because of our slightly casual treatment of vertical advection, but nonetheless it is apparent that the quasi-geostrophic potential vorticity equation is an approximation to the true potential vorticity.

The terms may all be written in terms of a streamfunction (see page 95 of *Essentials*):

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}, \qquad \zeta_g = \nabla^2 \psi, \qquad b' = f_0 \partial \psi / \partial z. \tag{V.74}$$

Thus, we have

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0, \qquad q = \nabla^2 \psi + f + f_0^2 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial \psi}{\partial z}\right). \tag{V.75}$$

where

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + \boldsymbol{u} \cdot \nabla q = \frac{\partial q}{\partial t} + J(\psi, q). \tag{V.76}$$

where $J(\psi, q) = \partial \psi/\partial x \, \partial q/\partial y - \partial \psi/\partial y \, \partial q/\partial x$.

V.5.2 QG from the Momentum Equation

In this section we derive the quasi-geostrophic equations by making the main approximations in the momentum equation, rather than in the vorticity equation. We carry through the derivation fairly systematically in the shallow water equations and then, more briefly, in the stratified Boussinesq equations. We will use the dimensional form of the equations in Cartesian co-ordinates on the β-plane. The three main assumptions we make are

(i) That the Rossby number is small and the flow is in near geostrophic balance.
(ii) That variations in the Coriolis parameter are small: $|\beta y| \ll f_0$.
(iii) That scales of motion are similar to the deformation radius. In the shallow water equations this implies that variations in the height field are small compared to the mean height, and in the stratified equations it implies that the variations in buoyancy are small compared to the mean state. (For more discussion of this, see Chapter 6 in *Essentials*.)

The quasi-geostrophic shallow water equations

With our usual notation the shallow-water momentum equations may be written as

$$\frac{\mathrm{D}u}{\mathrm{D}t} - fv = -g\frac{\partial h}{\partial x}, \tag{V.77a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + fu = -g\frac{\partial h}{\partial y}, \tag{V.77b}$$

where $f = f_0 + \beta y$. .We may decompose the velocity field into a geostrophic flow, $\boldsymbol{u}_g$ and an ageostrophic flow, $\boldsymbol{u}_a$, such that $\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a$. Since variations in the Coriolis parameter are small, we shall define the geostophic flow to satisfy

$$-f_0 v_g = -g\frac{\partial h}{\partial x}, \qquad \text{and} \qquad f_0 u_g = -g\frac{\partial h}{\partial y}, \tag{V.78a,b}$$

Subtracting (V.78) from (V.77) gives, without approximation,

$$\frac{\mathrm{D}u}{\mathrm{D}t} - \beta y v_g - (f_0 + \beta y)v_a = 0, \tag{V.79a}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \beta y u_g + (f_0 + \beta y)u_a = 0, \tag{V.79b}$$

The velocity components in the material derivative are the full ones, that is $u_g + u_a$ and similarly for v. Equation (V.79) is exactly equivalent to (V.77).

We now invoke approximations (1) and (2) above. Thus, we ignore the ageostrophic velocity where it appears alongside a geostrophic velocity (since $|\boldsymbol{u}_g| \gg |\boldsymbol{u}_a|$), and we ignore βy where it appears alongside f_0. We then obtain

$$\frac{\mathrm{D}u_g}{\mathrm{D}t} - \beta y v_g - f_0 v_a = 0, \tag{V.80a}$$

$$\frac{\mathrm{D}v_g}{\mathrm{D}t} + \beta y u_g + f_0 u_a = 0, \tag{V.80b}$$

The advecting velocity in the material derivatives is by the geostrophic velocity. These equations are sometimes called the *quasi-geostrophic momentum equations.* The equations are not closed — there are two equations and four unknowns. Note also that the geostrophic velocity is divergence-free, but the ageostrophic velocity is not; that is

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = -\frac{\partial w}{\partial z}, \qquad \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0. \tag{V.81}$$

In the shallow water equations it is more usual to write the mass continuity equation in the form

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\nabla\cdot\boldsymbol{u} = 0, \tag{V.82}$$

where $\nabla\cdot\boldsymbol{u} = \partial u_a/\partial x + \partial v_a/\partial y$ since only the ageostrophic flow has a divergence.

If we cross differentiate (V.80a) and (V.80b) we obtain, after just a little algebra, the vorticity equation,

$$\frac{\mathrm{D}\zeta_g}{\mathrm{D}t} + \beta v_g = -f_0\nabla\cdot\boldsymbol{u}_a \tag{V.83}$$

where ζ_g is the vorticity of the geostrophic flow, $\partial v_g/\partial x - \partial u_g/\partial y$. This equation is essentially the same as (5.22). If we now use (V.82) to evaluate the divergence term we obtain

$$\frac{\mathrm{D}\zeta_g}{\mathrm{D}t} + \beta v_g = \frac{f_0}{h}\frac{\mathrm{D}h}{\mathrm{D}t}. \tag{V.84}$$

We now use assumption (3), that the variations in h are small, and replace h by H, the mean fluid depth, in the denominator of the term on the right-hand side, giving,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\zeta_g - f_0\frac{h}{H}\right) + \beta v_g = 0. \tag{V.85}$$

The final step in the derivation is realise that the height and the velocity, and thus also the vorticity, and relatd by geostrophic balance, (V.78). We may then conveniently express all of the variables in terms of a streamfunction,

$$h = \frac{f_0\psi}{g}, \quad u_g = -\frac{\partial\psi}{\partial y}, \quad v_g = \frac{\partial\psi}{\partial x}, \quad \zeta_g = \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}. \tag{V.86}$$

Equation (V.83) becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\nabla^2\psi - \frac{1}{L_d^2}\psi + \beta y\right) = 0. \tag{V.87}$$

This is the quasi-geostrophic potential vorticity equation for the shallow water equations.

Continuously Stratified, Boussinesq Case

The continuously stratified derivation follows in almost the same way. The vorticity equation is the same as (V.83), namely

$$\frac{D\zeta_g}{Dt} + \beta v_g = -f_0 \nabla \cdot \boldsymbol{u}_a, \tag{V.88}$$

where $\nabla \cdot \boldsymbol{u}_a = \partial u_a/\partial x + \partial v_a/\partial y$. We use the mass continuity equation, $\nabla \cdot \boldsymbol{u}_a = -\partial w/\partial z$ and (V.88) becomes

$$\frac{D\zeta_g}{Dt} + \beta v_g = f_0 \frac{\partial w}{\partial z}. \tag{V.89}$$

The advecting flow is just the horizontal geostrophic flow, and this equation is then the same as (5.59).

To eliminate the vertical velocity we begin with the thermodynamic equation, namely,

$$\frac{Db}{Dt} = 0 \tag{V.90}$$

where b is the buoyancy, $b = -g\,\delta\rho/\rho_0$ We now divide the buoyancy into a mean state and a variation about this:

$$b(x, y, z, t) = \tilde{b}(z) + b'(x, y, z, t), \tag{V.91}$$

In the shallow water derivation we assumed that variations in the height field were small compared to the mean height; we now assume that variations in buoyancy are small compared to the mean buoyancy and (V.90) becomes

$$\frac{Db'}{Dt} + wN^2 = 0, \tag{V.92}$$

where $N^2 = \partial\tilde{b}/\partial z$. The advection in the material derivative are taken to be only by the geostrophic velocity; there is therefore no vertical component, but there is a vertical advection of the mean state, wince $wN^2 = w\partial\tilde{b}/\partial z$. Given this, (V.92) is the same as (5.60).

We now eliminate vertical velocities between (V.89) and (V.92) to obtain, after some algebra,

$$\frac{D}{Dt}\left(\zeta_g + f_0 \frac{\partial b}{\partial z}\right) + \beta v_g = 0. \tag{V.93}$$

The final step, analogous to that in the shallow water derivation, is to realize that the velocity and buoyancy field are related through the thermal wind equation and therefore that both can be related to a streamfunction:

$$b = f_0 \frac{\partial \psi}{\partial z}, \quad u_g = -\frac{\partial \psi}{\partial y}, \quad v_g = \frac{\partial \psi}{\partial x}, \quad \zeta_g = \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{V.94}$$

Using (V.94) in (V.93) we obtain the Boussinesq quasi-geostrophic potential vorticity equation, namely

$$\frac{D}{Dt}\left(\nabla^2 \psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta y\right) = 0. \tag{V.95}$$

Notes and References

This material supplements that in *Essentials of Atmospheric and Oceanic Dynamics*.

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